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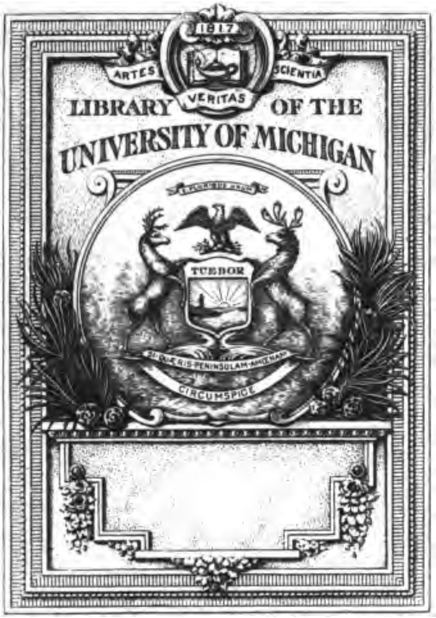
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DIFFERENTIAL AND INTEGRAL CALCULUS

WITH EXAMPLES AND APPLICATIONS

BY

bbott

GEORGE A. OSBORNE, S.B.

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PREFACE

In the original work, the author endeavored to prepare a textbook on the Calculus, based on the method of limits, that should be within the capacity of students of average mathematical ability, and yet contain all that is essential to a working knowledge of the subject.

In the revision of the book the same object has been kept in view. Most of the text has been rewritten, the demonstrations have been carefully revised, and for the most part new examples have been substituted for the old. There has been some rearrangement of subjects in a more natural order.

In the Differential Calculus, illustrations of the "derivative" have been introduced in Chapter II., and applications of differentiation will be found also among the examples in the chapter immediately following.

Chapter VII., on Series, is entirely new. In the Integral Calculus, immediately after the integration of standard forms, Chapter XXI. has been added, containing simple applications of integration.

In both the Differential and Integral Calculus, examples illustrating applications to Mechanics and Physics will be found, especially in Chapter X. of the Differential Calculus, on Maxima and Minima, and in Chapter XXXII. of the Integral Calculus. The latter chapter has been prepared by my colleague, Assistant Professor N. R. George, Jr.

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DIFFERENTIAL CALCULUS

CHAPTER I

FUNCTIONS

1. Variables and Constants. A quantity which may assume an unlimited number of values is called a *variable*.

A quantity whose value is unchanged is called a *constant*.

For example, in the equation of the circle

$$x^2 + y^2 = a^2,$$

x and y are variables, but a is a constant. For as the point whose coördinates are x, y, moves along the curve, the values of x and y are continually changing, while the value of the radius a remains unchanged.

Constants are usually denoted by the first letters of the alphabet, a, b, c, α, β, γ, etc.

Variables are usually denoted by the last letters of the alphabet, x, y, z, ϕ, ψ, etc.

2. Function. When one variable quantity so depends upon another that the value of the latter determines that of the former, the former is said to be a *function* of the latter.

For example, the area of a square is a function of its side; the volume of a sphere is a function of its radius; the sine, cosine, and tangent are functions of the angle; the expressions

$$x^2, \ \log(x^2+1), \ \sqrt{x(x+1)},$$

are functions of x.

A quantity may be a function of two or more variables. For example, the area of a rectangle is a function of two adjacent sides; either side of a right triangle is a function of the two other sides; the volume of a rectangular parallelopiped is a function of its three dimensions.

The expressions

$$x^2 + xy + y^2, \quad \log(x^2 + y^2), \quad a^{x+y},$$

are functions of x and y.

The expressions

$$xy + yz + zx, \quad \sqrt{\frac{x+y}{z}}, \quad \log(x^2 + y - z),$$

are functions of x, y, and z.

3. Dependent and Independent Variables. If y is a function of x, as in the equations

$$y = x^2, \quad y = \tan 4x, \quad y = e^x + 1,$$

x is called the *independent* variable, and y the *dependent* variable.

It is evident that when y is a function of x, x may be also regarded as a function of y, and the positions of dependent and independent variables reversed. Thus, from the preceding equations,

$$x = \sqrt{y}, \quad x = \tfrac{1}{4}\tan^{-1}y, \quad x = \log_e(y-1).$$

In equations involving more than two variables, as

$$z + x - y = 0, \quad w + wz + zx + y = 0,$$

one must be regarded as the dependent variable, and the others as independent variables.

4. Algebraic and Transcendental Functions. An *algebraic* function is one that involves only a finite number of the operations of addition, subtraction, multiplication, division, involution and evolution with constant exponents.* All other functions are called *transcendental* functions. Included in this class are *exponential, logarithmic, trigonometric* or *circular*, and *inverse trigonometric*, functions.

NOTE. — The term "hyperbolic functions" is applied to certain forms of exponential functions. (See Art. 277.)

---

* A more general definition of Algebraic Function is, a function whose relation to the variable is expressed by an algebraic equation.

5. Rational Functions. A polynomial involving only positive integral powers of x, is called an *integral function* of x; as, for example,

$$2+x-4x^2+3x^4.$$

A *rational fraction* is a fraction whose numerator and denominator are integral functions of the variable; as, for example,

$$\frac{3x^3+2x-1}{x^4+x^2-2x}.$$

A *rational function* of x is an algebraic function involving no fractional powers of x or of any function of x.

The most general form of such a function is the sum of an integral function and a rational fraction; as, for example,

$$2x^2+x-1+\frac{3x^3-2x}{x^3-2x^2+1}.$$

6. Explicit and Implicit Functions. When one quantity is expressed directly in terms of another, the former is said to be an *explicit* function of the latter.

For example, y is an explicit function of x in the equations

$$y=x^2+2x, \qquad y=\sqrt{x^2+1}.$$

When the relation between y and x is given by an equation containing these quantities, but not solved with reference to y, y is said to be an *implicit* function of x, as in the equations

$$axy+bx+cy+d=0, \qquad y+\log y=x.$$

Sometimes, as in the first of these equations, we can solve the equation with reference to y, and thus change the function from implicit to explicit. Thus we find from this equation,

$$y=-\frac{bx+d}{ax+c}.$$

7. Single-valued and Many-valued Functions. In the equation

$$y=x^2-2x,$$

for every value of x, there is one and only one value of y.

Expressing x in terms of y, we have

$$x=1\pm\sqrt{y+1}.$$

Here each value of y determines *two* values of x. In the former case, y is a *single-valued function* of x.

In the latter case, x is a *two-valued function* of y.

An n-valued function of a variable x is a function that has n values corresponding to each value of x.

The inverse trigonometric function, $\tan^{-1} x$, has an unlimited number of values for each value of x.

8. Notation of Functions. The symbols $F(x), f(x), \phi(x), \psi(x)$, and the like, are used to denote functions of x. Thus instead of "y is a function of x," we may write

$$y = f(x), \quad \text{or} \quad y = \phi(x).$$

A functional symbol occurring more than once in the same problem or discussion is understood to denote the same function or operation, although applied to different quantities. Thus if

$$f(x) = x^2 + 5, \quad \dots\dots\dots \quad (1)$$

then
$$f(y) = y^2 + 5, \qquad f(a) = a^2 + 5,$$
$$f(a+1) = (a+1)^2 + 5 = a^2 + 2a + 6,$$
$$f(2) = 2^2 + 5 = 9, \qquad f(1) = 6.$$

In all these expressions $f(\ )$ denotes the same operation as defined by (1); that is, the operation of squaring the quantity and adding 5 to the result.

Functions of two or more variables are expressed by commas between the variables.

Thus if
$$f(x, y) = x^2 + 3xy - y^2,$$

then
$$f(a, b) = a^2 + 3ab - b^2. \qquad f(b, a) = b^2 + 3ba - a^2.$$
$$f(3, 2) = 3^2 + 3 \cdot 3 \cdot 2 - 2^2 = 23. \qquad f(a, 0) = a^2.$$

If
$$\phi(x, y, z) = x^3 + yz - y^2 + 2,$$

then
$$\phi(3, 1, -1) = 3^3 + 1(-1) - 1^2 + 2 = 27;$$
$$\phi(a, b, 0) = a^3 - b^2 + 2; \qquad \phi(0, 0, 0) = 2.$$

9. Inverse Function. If y is a given function of x, represented by

$$y = \phi(x), \quad \dots\dots\dots \quad (1)$$

and if from this relation we express x in terms of y, so that

$$x = \psi(y), \quad \dots\dots\dots \quad (2)$$

then each of the functions ϕ and ψ is said to be the inverse of the other.

For example, if $y = x^3 = \phi(x)$,

then $x = \sqrt[3]{y} = \psi(y)$.

Here ψ, the *cube root* function, is the inverse of ϕ, the *cube* function.

If $y = a^x = \phi(x)$,

then $x = \log_a y = \psi(y)$.

Here ψ, the *logarithmic* function, is the inverse of ϕ, the *exponential* function.

Again, suppose

$$y = \frac{x+2}{1-x} = \phi(x). \quad \dots\dots \quad (3)$$

From this we derive

$$x = \frac{y-2}{y+1} = \psi(y). \quad \dots\dots\dots \quad (4)$$

Here ψ as defined by (4) is the inverse of ϕ as defined by (3).

The notation ϕ^{-1} is often employed for the inverse function of ϕ.

Thus, if $y = \phi(x)$, $x = \phi^{-1}(y)$.

If $y = f(x)$, $x = f^{-1}(y)$.

The student is already familiar with this notation for the inverse trigonometric functions.

If $y = \sin x$, $x = \sin^{-1} y$.

EXAMPLES

1. Given $2x^2 - 2xy + y^2 = a^2$;

change y from an implicit to an explicit function.

Ans. $y = x \pm \sqrt{a^2 - x^2}$.

2. Given $\sin(x-y) = m \sin y$;

change y from an implicit to an explicit function.

Ans. $y = \tan^{-1}\dfrac{\sin x}{m+\cos x}$.

3. Given $f(x) = 2x^3 - 3x^2 + x + 2$;

find $f(1),\ f(2),\ f(\tfrac{1}{2}),\ f(-1),\ f(0)$.

Show that $f(x+1) - f(x) = 6x^2$,

$$f(x+h) = f(x) + (6x^2 - 6x + 1)h + (6x-3)h^2 + 2h^3.$$

4. Given $F(x) = (x^2-1)^2$;

show that $F(x+1) - F(x-1) = 8x^3$.

5. Given $f(x) = \dfrac{ae^x + be^{-x}}{a+b}$; find $f(0),\ f(x) + f(-x)$.

Show that $f(2x) - f(-2x) = [f(x)]^2 - [f(-x)]^2$.

6. If $\phi(\theta) = e^\theta,\ \phi(a+b) = \phi(a)\,\phi(b)$.

Show that the same relation holds for the function

$$\psi(\theta) = \cos\theta + \sqrt{-1}\sin\theta,$$

giving $\psi(a+b) = \psi(a)\psi(b)$.

7. If $f(x) = \dfrac{3x-2}{5x-3}$,

show that the inverse function is of the same form.

8. If $\phi(x) = \dfrac{bx+d}{ax-c}$, find the inverse function of ϕ.

Compare the two functions when $b = c$.

9. If $f(x) = \log_a(x + \sqrt{x^2-1})$,

show that $f^{-1}(x) = \dfrac{a^x + a^{-x}}{2}$.

10. If $f(x, y) = ax^2 + 2bxy + cy^2$, find $f(1, 2), f(y, -x)$.

Show that

$$f(x+h, y+k) = f(x, y) + 2(ax+by)h + 2(bx+cy)k + f(h, k).$$

11. Given
$$\phi(m, n) = \frac{\underline{|m+n}}{\underline{|m}\,\underline{|n}},$$
where m, n, are positive integers; show that
$$\phi(m, n+1) + \phi(m+1, n) = \phi(m+1, n+1).$$

12. Given
$$f(x, y, z) = \begin{vmatrix} x, & y, & z \\ z, & x, & y \\ y, & z, & x \end{vmatrix};$$
show that $f(y+z, z+x, x+y) = 2f(x, y, z)$.

CHAPTER II

LIMIT. INCREMENT. DERIVATIVE

10. Limit. When the successive values of a variable x approach nearer and nearer a fixed value a, in such a way that the difference $x - a$ becomes and remains as small as we please, the value a is called the *limit* of the variable x.

The student is supposed to be already somewhat familiar with the meaning of this term, of which the following illustrations may be mentioned.

The limit of the value of the recurring decimal .3333 ..., as the number of decimal places is indefinitely increased, is $\frac{1}{3}$.

The limit of the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, as the number of terms is indefinitely increased, is 2.

The limit of the fraction $\dfrac{x^3 - a^3}{x - a}$, as x approaches a, is $3a^2$.

The circle is the limit of a regular polygon, as the number of sides is indefinitely increased.

The limit of the fraction $\dfrac{\sin\theta}{\theta}$, as θ approaches zero, is 1, provided θ is expressed in circular measure.

11. Notation of Limit. The following notation will be used: "$\text{Lim}_{x=a}$" denotes "The limit, as x approaches a, of."

For example, $$\text{Lim}_{x=a}\frac{x^2 - a^2}{x^2 - ax} = 2.$$

$$\text{Lim}_{h=0}(2x^2 - hx + h^2) = 2x^2.$$

12. Some Special Limits. There are two important limits required in the following chapter.

(*a*) $\text{Lim}_{\theta=0}\dfrac{\sin\theta}{\theta}$, *$\theta$ being in circular measure.*

Let the angle $AOA' = 2\theta$, and let a be the radius of the arc ACA'

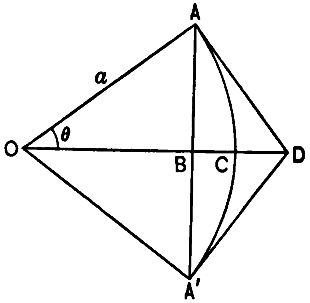
From geometry, $ABA' < ACA'$;

that is, $2a \sin\theta < 2a\theta$, $\frac{\sin\theta}{\theta} < 1$. (1)

Also from geometry, $ACA' < ADA'$;

that is, $2a\theta < 2a\tan\theta$, $\frac{\sin\theta}{\cos\theta} > \theta$,

$\frac{\sin\theta}{\theta} > \cos\theta$. . . . . . . (2)



Hence by (1) and (2), $\frac{\sin\theta}{\theta}$ is intermediate in value between 1 and $\cos\theta$. As θ approaches zero, $\cos\theta$ approaches 1.

Hence $\text{Lim}_{\theta=0}\frac{\sin\theta}{\theta} = 1$.

The student will do well to compare the corresponding values of θ and $\sin\theta$, taken from the tables, for angles of 5°, 1°, and 10′.

Angle	θ	$\sin\theta$
5°	$\frac{\pi}{36} = .0872665$	.0871557
1°	$\frac{\pi}{180} = .0174533$	.0174524
10′	$\frac{\pi}{1080} = .0029089$	.0029089

(*b*) $\text{Lim}_{z=\infty}\left(1 + \frac{1}{z}\right)^z$. Before deriving this limit let us compute the value of the expression for increasing values of z. Thus,

$$\begin{aligned}
(1 + \tfrac{1}{2})^2 &= 2.25\\
(1 + \tfrac{1}{5})^5 &= 2.48832\\
(1 + \tfrac{1}{10})^{10} &= 2.59374\\
(1.01)^{100} &= 2.70481\\
(1.001)^{1000} &= 2.71692\\
(1.0001)^{10000} &= 2.71815\\
(1.00001)^{100000} &= 2.71827\\
(1.000001)^{1000000} &= 2.71828
\end{aligned}$$

The required limit will be found to agree to five decimals with the last number, 2.71828.

By the Binomial Theorem,

$$\left(1+\frac{1}{z}\right)^z = 1 + z\frac{1}{z} + \frac{z(z-1)}{\lfloor\underline{2}}\left(\frac{1}{z}\right)^2 + \frac{z(z-1)(z-2}{\lfloor\underline{3}}\left(\frac{1}{z}\right)^3 + \cdots,$$

which may be written

$$\left(1+\frac{1}{z}\right)^z = 1 + 1 + \frac{1-\frac{1}{z}}{\lfloor\underline{2}} + \frac{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)}{\lfloor\underline{3}} + \cdots.$$

When z increases, the fractions $\frac{1}{z}$, $\frac{2}{z}$, etc., approach zero, and we have

$$\text{Lim}_{z=\infty}\left(1+\frac{1}{z}\right)^z = 1 + 1 + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} + \cdots.^*$$

This quantity is usually denoted by e, so that

$$e = 1 + \frac{1}{1} + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} + \cdots.$$

The value of e can be easily calculated to any desired number of decimals by computing the values of the successive terms of this series. For seven decimal places the calculation is as follows:

```
      1.
  2)  1.
  3)   .5
  4)   .166666667
  5)   .041666667
  6)   .008333333
  7)   .001388889
  8)   .000198413
  9)   .000024802
 10)   .000002756
 11)   .000000276
       .000000025
 ----------------
 e =  2.7182818...
```

This quantity e is the base of the Napierian logarithms.

---

[*] For a rigorous derivation of this limit, the student is referred to more extensive treatises on the Differential Calculus.

13. Increments. An increment of a variable quantity is any addition to its value, and is denoted by the symbol Δ written before this quantity. Thus Δx denotes an increment of x, Δy, an increment of y.

For example, if we have given

$$y = x^2,$$

and assume $x = 10$, then if we increase the value of x by 2, the value of y is increased from 100 to 144, that is, by 44.

In other words, if we assume the increment of x to be $\Delta x = 2$, we shall find the increment of y to be $\Delta y = 44$.

If an increment is *negative*, there is a *decrease* in value.

For example, calling $x = 10$ as before, in $y = x^2$,

$$\text{if} \quad \Delta x = -2, \qquad \text{then} \quad \Delta y = -36.$$

14. Derivative. With the same equation,

$$y = x^2,$$

and the same initial value of x,

$$x = 10,$$

let us calculate the values of Δy corresponding to different values of Δx. We thus find results as in the following table.

If $\Delta x =$	then $\Delta y =$	and $\frac{\Delta y}{\Delta x} =$
3.	69.	23.
2.	44.	22.
1.	21.	21.
0.1	2.01	20.1
0.01	0.2001	20.01
0.001	0.020001	20.001
h	$20h + h^2$	$20 + h$

The third column gives the value of the ratio between the increments of x and of y.

It appears from the table that, as Δx diminishes and approaches zero, Δy also diminishes and approaches zero.

The ratio $\frac{\Delta y}{\Delta x}$ diminishes, but instead of approaching zero, approaches 20 as its limit.

This limit of $\frac{\Delta y}{\Delta x}$ is called the *derivative* of y with respect to x, and is denoted by $\frac{dy}{dx}$. In this case, when $x = 10$, the derivative $\frac{dy}{dx} = 20$.

It will be noticed that the value 20 depends partly on the function $y = x^2$, and partly on the initial value 10 assigned to x.

Without restricting ourselves to any one initial value, we may obtain $\frac{dy}{dx}$ from $y = x^2$.

Increase x by Δx. Then the new value of y will be

$$y' = (x + \Delta x)^2;$$

therefore, $$\Delta y = y' - y = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2.$$

Dividing by Δx, $$\frac{\Delta y}{\Delta x} = 2\,x + \Delta x.$$

The limit of this, when Δx approaches zero, is $2x$.

Hence, $$\frac{dy}{dx} = 2\,x.$$

The *derivative* of a function may then be defined as *the limiting value of the ratio of the increment of the function to the increment of the variable, as the latter increment approaches zero.*

It is to be noticed that $\frac{dy}{dx}$ is not here defined as a *fraction*, but as a single symbol denoting the *limit* of the fraction $\frac{\Delta y}{\Delta x}$. The student will find as he advances that $\frac{dy}{dx}$ has many of the properties of an ordinary fraction.

The derivative is sometimes called the *differential coefficient.*

15. General Expression for Derivative. In general, let

$$y = f(x).$$

Increase x by Δx, and we have the new value of y,

$$y' = f(x + \Delta x).$$

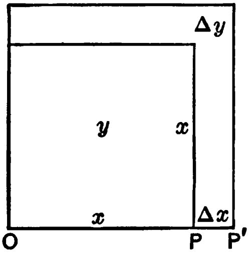
$$\Delta y = y' - y = f(x + \Delta x) - f(x),$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

$$\frac{dy}{dx} = \text{Lim}_{\Delta x = 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Geometrical Illustration. The process of finding the derivative from $y = x^2$, may be illustrated by a square.

Let x be the length of the side OP, and y the area of the square on OP.



That is, y is the number of square units corresponding to the linear unit of x.

When the side is increased by PP', the area is increased by the space between the squares.

That is, $\Delta y = 2x\Delta x + (\Delta x)^2$, $\frac{\Delta y}{\Delta x} = 2x + \Delta x$,

$$\frac{dy}{dx} = \text{Lim}_{\Delta x = 0} \frac{\Delta y}{\Delta x} = 2x.$$

16. From the definition of the derivative we have the following process for obtaining it:

(*a*) Increase x by Δx, and by substituting $x + \Delta x$ for x, determine $y + \Delta y$, the new value of y.

(*b*) Find Δy by subtracting the initial value of y from the new value.

(*c*) Divide by Δx, giving $\frac{\Delta y}{\Delta x}$.

(*d*) Determine the limit of $\frac{\Delta y}{\Delta x}$, as Δx approaches zero. This limit is the derivative $\frac{dy}{dx}$.

Apply this process to the following examples.

EXAMPLES

1. $y = 2x^3 - 6x + 5$.

Increasing x by Δx, we have

$$y + \Delta y = 2(x + \Delta x)^3 - 6(x + \Delta x) + 5;$$

therefore, $\Delta y = 2(x + \Delta x)^3 - 6(x + \Delta x) + 5 - 2x^3 + 6x - 5$

$$= (6x^2 - 6)\Delta x + 6x(\Delta x)^2 + 2(\Delta x)^3.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x}=6x^2-6+6x\Delta x+2(\Delta x)^2.$$

$$\frac{dy}{dx}=\text{Lim}_{\Delta x=0}\frac{\Delta y}{\Delta x}=6x^2-6.$$

2. $$y=\frac{x}{x+1}.$$

$$y+\Delta y=\frac{x+\Delta x}{x+\Delta x+1}.$$

$$\Delta y=\frac{x+\Delta x}{x+\Delta x+1}-\frac{x}{x+1}=\frac{\Delta x}{(x+\Delta x+1)(x+1)}.$$

$$\frac{\Delta y}{\Delta x}=\frac{1}{(x+\Delta x+1)(x+1)}.$$

$$\frac{dy}{dx}=\text{Lim}_{\Delta x=0}\frac{\Delta y}{\Delta x}=\frac{1}{(x+1)^2}.$$

3. $$y=\sqrt{x}.$$

$$y+\Delta y=\sqrt{x+\Delta x},$$

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x},$$

$$\frac{\Delta y}{\Delta x}=\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}.$$

The limit of this takes the indeterminate form $\frac{0}{0}$. But by rationalizing the numerator, we have

$$\frac{\Delta y}{\Delta x}=\frac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}.$$

$$\frac{dy}{dx}=\text{Lim}_{\Delta x=0}\frac{\Delta y}{\Delta x}=\frac{1}{2\sqrt{x}}.$$

4. $y=x^4-2x^2+3x-4$, $\qquad \frac{dy}{dx}=4x^3-4x+3.$

5. $y=(x-a)^3$, $\qquad \frac{dy}{dx}=3(x-a)^2.$

6. $y=(t+2)(3-2t)$, $\qquad \dfrac{dy}{dt}=-4t-1.$

7. $y=\dfrac{1}{x^3}$, $\qquad \dfrac{dy}{dx}=-\dfrac{3}{x^4}.$

8. $y=\dfrac{mx}{n-x}$, $\qquad \dfrac{dy}{dx}=\dfrac{mn}{(n-x)^2}.$

9. $x=\dfrac{2y-5}{y+2}$, $\qquad \dfrac{dx}{dy}=\dfrac{9}{(y+2)^2}.$

10 $y=\dfrac{x^2+a^2}{x+a}$, $\qquad \dfrac{dy}{dx}=1-\dfrac{2a^2}{(x+a)^2}$

11. $x=\dfrac{t}{(t-1)^2}$, $\qquad \dfrac{dx}{dt}=-\dfrac{t+1}{(t-1)^3}.$

12. $y=\sqrt{x+2}$, $\qquad \dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x+2}}.$

13. $y=x^{\frac{3}{2}}$, $\qquad \dfrac{dy}{dx}=\dfrac{3x^{\frac{1}{2}}}{2}.$

14. $y=\sqrt{a^2-x^2}$, $\qquad \dfrac{dy}{dx}=-\dfrac{x}{\sqrt{a^2-x^2}}.$

15. $x=\dfrac{1}{t^{\frac{1}{2}}}$, $\qquad \dfrac{dx}{dt}=-\dfrac{1}{2t^{\frac{3}{2}}}.$

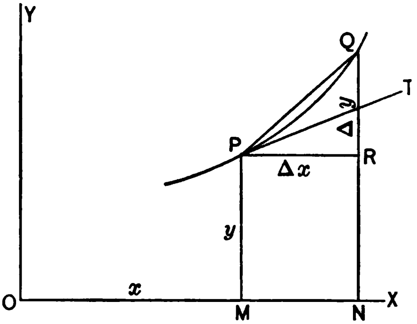
16. Show that the derivative of the area of a circle, with respect to its radius, is its circumference.

17. Show that the derivative of the volume of a sphere, with respect to its radius, is the surface of the sphere.

We shall now give some illustrations of the meaning of the derivative.

17. Direction of a Plane Curve. This is one of the simplest and most useful interpretations of the derivative.

Let P be a point in a curve determined by its equation $y = f(x)$, and PT the tangent at P.



Let $OM = x$, $MP = y$.

If we give x the increment $\Delta x = MN$, y will have the increment $\Delta y = RQ$.

Draw PQ. Then

$$\tan RPQ = \frac{RQ}{PR} = \frac{\Delta y}{\Delta x}. \quad (1)$$

Now if Δx diminish and approach zero, Δy will also approach zero, the point Q will move along the curve towards P, and PQ will approach in direction PT as its limit. Taking the limit of each member of (1), we have

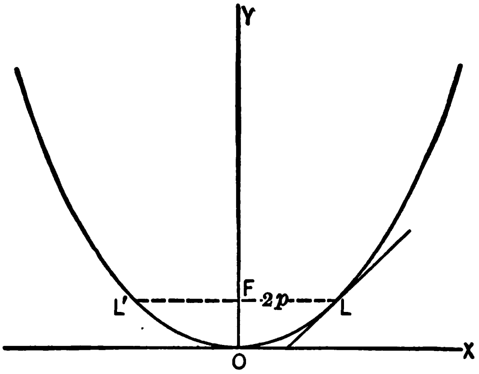
$$\tan TPR = \text{Lim}_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

That is, the derivative $\frac{dy}{dx}$, at any point of a curve, is the trigonometric tangent of the inclination to OX of the tangent line at that point.

This quantity is denoted by the term *slope*.

The *slope* of a straight line is the tangent of its inclination to the axis of X.

The *slope* of a plane curve at any point is the slope of its tangent at that point.



Thus, $\frac{dy}{dx}$, at any point of a curve, is the *slope* of the curve at that point.

For example, consider the parabola $x^2 = 4py$, $y = \dfrac{x^2}{4p}$.

The slope of the curve is $\dfrac{dy}{dx} = \dfrac{x}{2p}$.

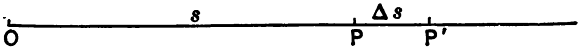
At O, where $x = 0$, the slope $= 0$, the direction being horizontal.

At L, where $x = 2p$, the slope $= 1$, corresponding to an inclination of 45° to the axis of X.

Beyond L the slope increases towards ∞, the inclination increasing towards the limit 90°.

For all points on the left of OY, x is negative, and hence the slope is negative, the corresponding inclinations to the axis of X being negative.

18. Velocity in Terms of a Variable t denoting Time. A body moves over the distance $OP = s$ in the time t, s being a function of t; it is required to express the velocity at the point P.



Let Δs denote the distance PP' traversed in the interval Δt. If the velocity were uniform during this interval, it would be equal to $\dfrac{\Delta s}{\Delta t}$.

For a variable velocity, $\dfrac{\Delta s}{\Delta t}$ is the average or mean velocity between P and P', and is more nearly equal to the velocity at P the less we make Δt.

That is, the velocity at $P = \text{Lim}_{\Delta t = 0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt}$.

If v denote this velocity, $v = \dfrac{ds}{dt}$.

Thus, $\dfrac{ds}{dt}$ is the rate of increase of s.

Similarly, $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are the rates of increase of x and y respectively.

19. Acceleration. The rate of increase of the velocity v is called *acceleration.*

If we denote this by α, we have by the preceding article,

$$\alpha = \frac{dv}{dt}.$$

For example, suppose a body moves so that

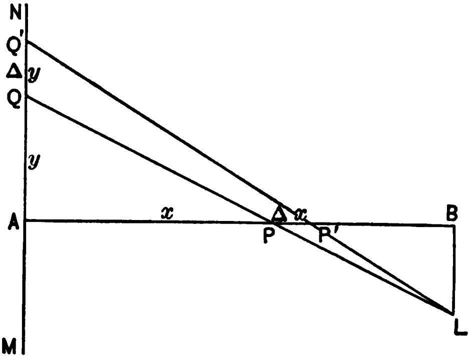
$$s = t^3.$$

Then the velocity, $$v = \frac{ds}{dt} = 3\,t^2,$$

and the acceleration, $$\alpha = \frac{dv}{dt} = 6\,t.$$

20. Rates of Increase of Variables. For further illustrations of the derivative, consider the two following problems:

Problem 1. A man walks across the street from A to B at a uniform rate of 5 feet per second. A lamp at L throws his shadow upon the wall MN. AB is 36 feet, and BL 4 feet. How fast is the shadow moving when he is 16 feet from A? When 26 feet? When 30 feet?



Let P and Q be simultaneous positions of man and shadow. Let $AP = x$, $AQ = y$.

Then $$\frac{y}{x} = \frac{BL}{PB} = \frac{4}{36 - x}, \quad y = \frac{4\,x}{36 - x}. \qquad (1)$$

When he walks from P to P', the shadow moves from Q to Q'. That is, when $\Delta x = PP'$, $\Delta y = QQ'$.

Let Δt be the interval of time corresponding to Δx and Δy.

Then we may write $$\frac{\Delta y}{\Delta x} = \frac{\dfrac{\Delta y}{\Delta t}}{\dfrac{\Delta x}{\Delta t}}. \qquad (2)$$

If now we suppose Δt to diminish indefinitely, Δy and Δx will also diminish indefinitely, and we have for the limits of the two members of (2),

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\text{rate of increase of } y}{\text{rate of increase of } x}. \qquad \text{Art. 18.}$$

That is, $\dfrac{\text{velocity of shadow at any point } Q}{\text{velocity of man}}=\dfrac{dy}{dx}.$

Finding the derivative of (1), we have

$$\frac{dy}{dx}=\frac{144}{(36-x)^2}. \qquad \text{See Ex. 8, Art. 16.}$$

Hence,

velocity of shadow at any point $Q=\dfrac{144}{(36-x)^2}$ (velocity of man)

$$=\frac{144}{(36-x)^2}\text{ (5 feet per second)}$$

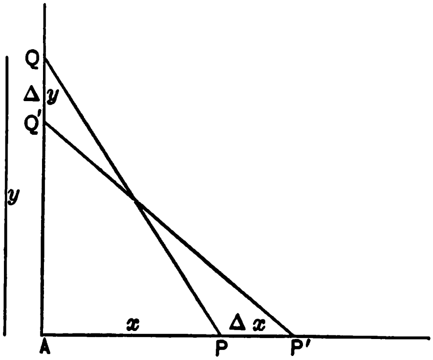
$$=\frac{720}{(36-x)^2}\text{ feet per second}$$

$= 1.8$ feet per second, when $x=16$;

$= 7.2$ feet per second, when $x=26$;

$= 20$ feet per second, when $x=30$.

Problem 2. The top of a ladder 20 feet long rests against a wall. The foot of the ladder is moved away from the wall at a uniform rate of 2 feet per second. How fast is the top moving, when the foot is 12 feet from the wall? When 16 feet from the wall?



Suppose PQ to be one position of the ladder.

Let

$$AP=x,\ AQ=y.$$

Then

$$y=\sqrt{400-x^2}. \qquad (3)$$

When the foot moves from P to P', the top moves from Q to Q'. That is, if $\Delta x = PP'$, $\Delta y = QQ'$.

In the same way as in Problem 1,

$$\frac{\Delta y}{\Delta x} = \frac{\dfrac{\Delta y}{\Delta t}}{\dfrac{\Delta x}{\Delta t}}.$$

And from this, $$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}};$$

that is, $$\frac{\text{velocity of top at } Q}{\text{velocity of foot}} = \frac{dy}{dx}.$$

From (3), $$\frac{dy}{dx} = \frac{-x}{\sqrt{400 - x^2}}.$$ See Ex. 14, Art. 16.

Hence,

$$\text{velocity of top at any point } Q = -\frac{x}{\sqrt{400 - x^2}} (\text{velocity of foot})$$

$$= -\frac{2x}{\sqrt{400 - x^2}} \text{ feet per second.}$$

The negative sign is explained by noticing from the figure that y decreases when x increases. Hence the rates of increase of x and y have different signs.

When $x = 12$, velocity of top $= -1\frac{1}{2}$ feet per second.

When $x = 16$, velocity of top $= -2\frac{2}{3}$ feet per second.

From these problems it appears that, while $\frac{\Delta y}{\Delta x}$ is the ratio between the increments of y and x, $\frac{dy}{dx}$ is the *ratio between the rates of increase of these variables.*

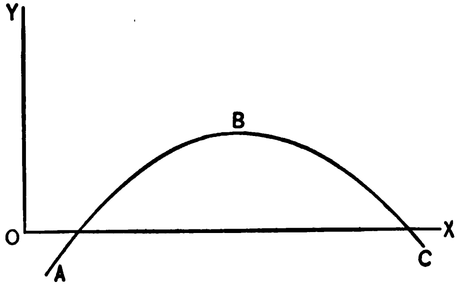
21. Increasing and Decreasing Functions. *If the derivative of a function of x is positive, the function increases when x increases; and if the derivative is negative, the function decreases when x increases.*

For if the derivative $\frac{dy}{dx}$, which is the ratio between the rates of increase of the variables (see conclusion of Art. 20), is positive, it follows that these rates must have the same sign; that is, y increases when x increases, and decreases when x decreases.

But if $\frac{dy}{dx}$ is negative, the rates must have different signs; that is, y decreases when x increases, and increases when x decreases.

This is also evident geometrically by regarding $\frac{dy}{dx}$ as the slope of a curve.

As we pass from A to B, y increases as x increases, but from B to C, y decreases as x increases.



Between A and B the slope $\frac{dy}{dx}$ is positive; between B and C, negative.

In the former case y is said to be an *increasing* function; in the latter case, a *decreasing* function.

For example, consider the function $y = x^3$, from which we find $\frac{dy}{dx} = 3x^2$.

Since $\frac{dy}{dx}$ is positive for all values of x, the function $y = x^3$ is an increasing function.

If we take $y = \frac{1}{x}$, we find $\frac{dy}{dx} = -\frac{1}{x^2}$.

Here we have a decreasing function with a negative derivative.

Another illustration is Ex. 1, Art. 16,

$$y = 2x^3 - 6x + 5, \quad \frac{dy}{dx} = 6(x^2 - 1).$$

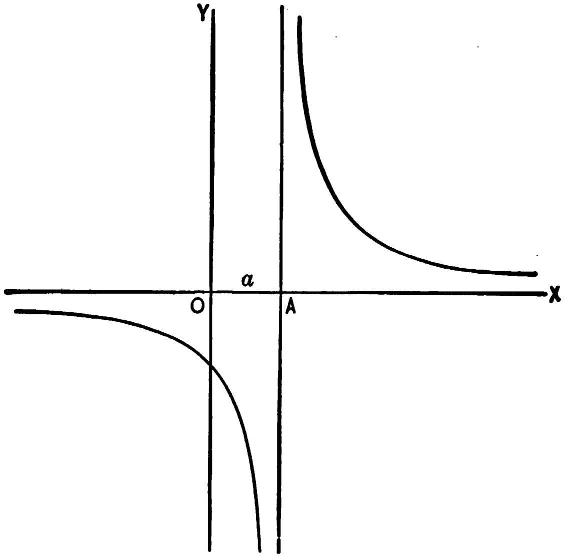
When x is numerically less than 1, y is a decreasing function.

When x is numerically greater than 1, y is an increasing function.

22. Continuous Function. A function, $y = f(x)$, is said to be *continuous* for a certain value x_0, of x, when $y_0 = f(x_0)$ is a definite quantity, and Δy_0 approaches zero as Δx_0 approaches zero, Δx_0 being positive or negative.

The latter condition is sometimes expressed, "when an infinitely small increment in x produces an infinitely small increment in y."

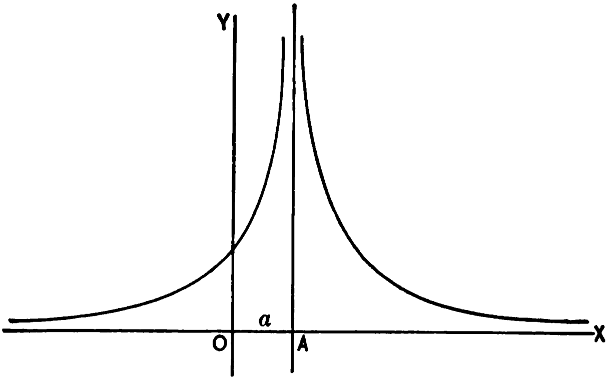
The most common case of discontinuity of the elementary functions (algebraic, exponential, logarithmic, trigonometric and inverse trigonometric, functions) is when the function is infinite.



For example, consider the function $y = \dfrac{1}{x-a}$, which is continuous for all values of x except $x = a$.

When $x = a$, $y = \infty$, that is, y can be made as great as we please by taking x sufficiently near a. Also when $x<a$, y is negative, and when $x>a$, y is positive.

There is a break in the curve when $x = a$, and the function is said to be *discontinuous* for the value $x = a$.



The function $y=\dfrac{1}{(x-a)^2}$ is likewise discontinuous when $x=a$.

This function being positive for all values of x, the two branches of the curve are above the axis of x.

Likewise the functions, $\tan x$, $\sec x$, are discontinuous when $x=\dfrac{\pi}{2}$.

In general, if $f(x)=\infty$, when $x=a$, there is a break in the curve $y=f(x)$ corresponding to $x=a$, and both the curve and the function are then discontinuous.

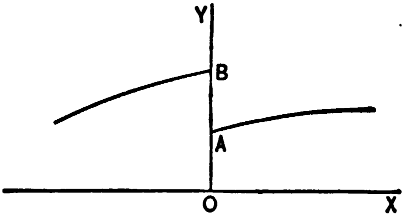
Another form of discontinuity is seen in the function $y=\dfrac{2^{\frac{1}{x}}+2}{2^{\frac{1}{x}}+1}$, when $x=0$.

Here y approaches two limits, according as x approaches zero through positive or negative values.

$$\text{Lim}_{x=+0}\frac{2^{\frac{1}{x}}+2}{2^{\frac{1}{x}}+1}=1. \qquad \text{Lim}_{x=-0}\frac{2^{\frac{1}{x}}+2}{2^{\frac{1}{x}}+1}=2.$$

We see that when $x=0$ the curve jumps from $y=2$ to $y=1$, that is from B to A.

The function is discontinuous for $x=0$.



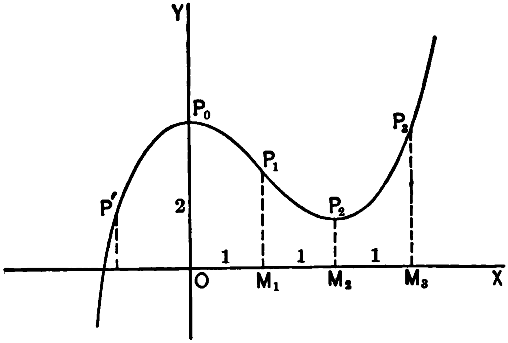
It is to be noticed that the definition of the derivative implies the continuity of the function. For $\frac{\Delta y}{\Delta x}$ cannot approach a limit, unless Δy approaches zero when Δx approaches zero.

The converse is not true. There are continuous functions which have no derivative, but they are never met with in ordinary practice.

EXAMPLES

1. The equation of a curve is $y = \frac{x^3}{3} - x^2 + 2$.

(*a*) Find its inclination to the axis of x, when $x = 0$, and when $x = 1$. *Ans.* 0° and 135°.



(*b*) Find the points where the curve is parallel to the axis of X. *Ans.* $x=0$ and $x=2$.

(*c*) Find the points where the slope is unity. *Ans.* $x = (1 \pm \sqrt{2})$.

(*d*) Find the point where the direction is the same as that at $x = 3$. *Ans.* $x = -1$.

2. In Problem 1, Art. 20, when will the velocity of the shadow be the same as that of the man? *Ans.* When $AP = 24$ ft.

When one quarter, and when nine times, that of the man?

Ans. When $AP = 12$ ft., and 32 ft.

3. A circular metal plate, radius r inches, is expanded by heat, the radius being expanded m inches per second. At what rate is the area expanded? *Ans.* $2\pi rm$ sq. in. per sec.

4. At what rate is the volume of a sphere increasing under the conditions of Ex. 3? *Ans.* $4\pi r^2 m$ cu. in. per sec.

5. The radius of a spherical soap bubble is increasing uniformly at the rate of $\frac{1}{10}$ inch per second. Find the rate at which the volume is increasing when the diameter is 3 inches.

Ans. $\frac{9\pi}{10} = 2.827$ cu. in. per sec.

6. In Exs. 5, 7, Art. 16, is y an increasing or a decreasing function? Is $\frac{3x+5}{x+1}$ an increasing or a decreasing function of x?

7. In the Example 1, above, for what values of x is y an increasing function of x, and for what values a decreasing function?

8. Find where the rate of change of the ordinate of the curve $y = x^3 - 6x^2 + 3x + 5$, is equal to the rate of change of the slope of the curve. *Ans.* $x = 5$ or 1.

9. When is the fraction $\frac{x^3}{x^2 + a^2}$ increasing at the same rate as x?

Ans. When $x^2 = a^2$.

10. If a body fall freely from rest in a vacuum, the distance through which it falls is approximately $s = 16\,t^2$, where s is in feet, and t in seconds. Find the velocity and acceleration. What is the velocity after 1 second? After 4 seconds? After 10 seconds?

Ans. 32, 128, and 320 ft. per sec.

CHAPTER III

DIFFERENTIATION

23. The process of finding the derivative of a given function is called *differentiation*. The examples in the preceding chapter illustrate the meaning of the derivative, but the elementary method of differentiation there used becomes very laborious for any but the simplest functions.

Differentiation is more readily performed by means of certain general rules or formulæ expressing the derivatives of the standard functions.

In these formulæ u and v will denote variable quantities, functions of x; and c and n constant quantities.

It is frequently convenient to write the derivative of a quantity u,

$$\frac{d}{dx}u \text{ instead of } \frac{du}{dx},$$

the symbol $\frac{d}{dx}$ denoting "derivative of."

Thus $\frac{d(u+v)}{dx}$, the derivative of $(u+v)$, may be written $\frac{d}{dx}(u+v)$.

24. Formulæ for Differentiation of Algebraic Functions.

$$\text{I.} \quad \frac{dx}{dx}=1.$$

$$\text{II.} \quad \frac{dc}{dx}=0.$$

$$\text{III.} \quad \frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}.$$

IV. $$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}.$$

V. $$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

VI. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

VII. $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

These formulæ express the following general rules of differentiation:

I. *The derivative of a variable with respect to itself is unity.*

II. *The derivative of a constant is zero.*

III. *The derivative of the sum of two variables is the sum of their derivatives.*

IV. *The derivative of the product of two variables is the sum of the products of each variable by the derivative of the other.*

V. *The derivative of the product of a constant and a variable is the product of the constant and the derivative of the variable.*

VI. *The derivative of a fraction is the derivative of the numerator multiplied by the denominator minus the derivative of the denominator multiplied by the numerator, this difference being divided by the square of the denominator.*

VII. *The derivative of any power of a variable is the product of the exponent, the power with exponent diminished by 1, and the derivative of the variable.*

25. Proof of I. This follows immediately from the definition of a derivative. For, since $\frac{\Delta x}{\Delta x} = 1$, its limit $\frac{dx}{dx} = 1$.

26. Proof of II. A constant is a quantity whose value does not vary.

Hence $\Delta c = 0$ and $\frac{\Delta c}{\Delta x} = 0$; therefore its limit $\frac{dc}{dx} = 0$.

27. Proof of III. Let $y = u + v$, and suppose that when x receives the increment Δx, u and v receive the increments Δu and Δv, respectively. Then the new value of y,

$$y + \Delta y = u + \Delta u + v + \Delta v,$$

therefore $$\Delta y = \Delta u + \Delta v.$$

Divide by Δx; then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Now suppose Δx to diminish and approach zero, and we have for the limits of these fractions,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

If in this we substitute for y, $u + v$, we have

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

It is evident that the same proof would apply to any number of terms connected by plus or minus signs. We should then have

$$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots.$$

28. Proof of IV. Let $y = uv$;

then $$y + \Delta y = (u + \Delta u)(v + \Delta v),$$

and $$\Delta y = (u + \Delta u)(v + \Delta v) - uv = v\Delta u + (u + \Delta u)\Delta v.$$

Divide by Δx;

then $$\frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + (u + \Delta u)\frac{\Delta v}{\Delta x}.$$

Now suppose Δx to approach zero, and, noticing that the limit of $u + \Delta u$ is u, we have

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx};$$

that is, $$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}.$$

29. Product of Several Factors. Formula IV. may be extended to the product of three or more factors. Thus we have

$$\frac{d}{dx}(uvw)=\frac{d}{dx}(uv\cdot w)=w\frac{d}{dx}(uv)+uv\frac{dw}{dx}$$

$$=w\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+uv\frac{dw}{dx}$$

$$=vw\frac{du}{dx}+uw\frac{dv}{dx}+uv\frac{dw}{dx}.$$

It appears from the preceding that the derivative of the product of two or three factors may be obtained by multiplying the derivative of each factor by all the others and adding the results.

This rule applies to the product of any number of factors. To prove this, we assume

$$\frac{d}{dx}\left(u_1u_2\cdots u_n\right)=u_2u_3\cdots u_n\frac{du_1}{dx}+u_1u_3u_4\cdots u_n\frac{du_2}{dx}+\cdots+u_1u_2\cdots u_{n-1}\frac{du_n}{dx}.$$

Then $\frac{d}{dx}\left(u_1u_2\cdots u_nu_{n+1}\right)=u_{n+1}\frac{d}{dx}\left(u_1u_2\cdots u_n\right)+u_1u_2\cdots u_n\frac{du_{n+1}}{dx}$

$$=u_2u_3\cdots u_{n+1}\frac{du_1}{dx}+u_1u_3u_4\cdots u_{n+1}\frac{du_2}{dx}+\cdots+u_1u_2\cdots u_{n-1}u_{n+1}\frac{du_n}{dx}$$

$$+u_1u_2\cdots u_n\frac{du_{n+1}}{dx}.$$

Thus it appears that if the rule applies to n factors, it holds also for $n+1$ factors, and is consequently applicable to any number of factors.

The derivative of the product of any number of factors is the sum of the products obtained by multiplying the derivative of each factor by all the other factors.

30. Proof of V. This is a special case of IV., $\frac{dc}{dx}$ being zero. But we may derive it independently thus:

$$y = cu,$$

$$y + \Delta y = c(u + \Delta u),$$

$$\Delta y = c\Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x},$$

$$\frac{dy}{dx} = c\frac{du}{dx}, \text{ or } \frac{d}{dx}(cu) = c\frac{du}{dx}.$$

31. Proof of VI. Let $y = \frac{u}{v}$.

Then
$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v};$$

therefore
$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{(v + \Delta v)v},$$

and
$$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{(v + \Delta v)v}.$$

Now suppose Δx to approach zero, and noticing that the limit of $v + \Delta v$ is v, we have

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Or we may derive VI. from IV. thus:

Since
$$y = \frac{u}{v},$$

therefore
$$yv = u.$$

By IV,

$$v\frac{dy}{dx}+y\frac{dv}{dx}=\frac{du}{dx},$$

$$v\frac{dy}{dx}=\frac{du}{dx}-\frac{u}{v}\frac{dv}{dx};$$

therefore

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

32. Proof of VII. *First*, suppose n to be a positive integer.

Let $$y=u^n,$$

then $$y+\Delta y=(u+\Delta u)^n,$$

and $$\Delta y=(u+\Delta u)^n-u^n.$$

Putting u' for $u+\Delta u$, we have

$$\Delta y=u'^n-u^n=(u'-u)\left(u'^{n-1}+u'^{n-2}u+u'^{n-3}u^2+\cdots+u^{n-1}\right).$$

that is, $$\Delta y=\Delta u\left(u'^{n-1}+u'^{n-2}u+u'^{n-3}u^2\cdots+u^{n-1}\right),$$

$$\frac{\Delta y}{\Delta x}=\left(u'^{n-1}+u'^{n-2}u+u'^{n-3}u^2\cdots+u^{n-1}\right)\frac{\Delta u}{\Delta x}.$$

Now let Δx diminish; then, u being the limit of u', each of the n terms within the parenthesis becomes u^{n-1}; therefore

$$\frac{dy}{dx}=nu^{n-1}\frac{du}{dx}.$$

Or it may be proved by regarding this as a special case of Art. 29 where u_1, u_2, $\cdots$ and u_n are each equal to u.

$$\text{Then }\frac{d}{dx}(u^n)=u^{n-1}\frac{du}{dx}+u^{n-1}\frac{du}{dx}+\cdots\text{ to } n \text{ terms}$$

$$=nu^{n-1}\frac{du}{dx}.$$

Second, suppose n to be a positive fraction, $\frac{p}{q}$.

Let $$y=u^{\frac{p}{q}},$$

then $$y^q=u^p;$$

therefore $$\frac{d}{dx}(y^q)=\frac{d}{dx}(u^p).$$

But we have already shown VII. to be true when the exponent is a positive integer; hence we may apply it to each member of this equation. This gives

$$qy^{q-1}\frac{dy}{dx}=pu^{p-1}\frac{du}{dx};$$

therefore $$\frac{dy}{dx}=\frac{p}{q}\frac{u^{p-1}}{y^{q-1}}\frac{du}{dx}.$$

Substituting for y, $u^{\frac{p}{q}}$, gives

$$\frac{dy}{dx}=\frac{p}{q}\frac{u^{p-1}}{u^{p-\frac{p}{q}}}\frac{du}{dx}=\frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx},$$

which shows VII. to be true in this case also.

Third, suppose n to be negative and equal to $-m$.

Let $$y=u^{-m}=\frac{1}{u^m};$$

by VI., $$\frac{dy}{dx}=\frac{-\frac{d}{dx}(u^m)}{u^{2m}}=\frac{-mu^{m-1}\frac{du}{dx}}{u^{2m}}=-mu^{-m-1}\frac{du}{dx}.$$

Hence, VII. is true in this case also.

EXAMPLES

Differentiate the following functions:

1. $y=x^4$.

$$\frac{dy}{dx}=\frac{d}{dx}(x^4).$$

If we apply VII., substituting $u=x$ and $n=4$, we have

$$\frac{d}{dx}(x^4)=4x^3\frac{dx}{dx}=4x^3. \quad \text{by I.}$$

Hence, $$\frac{dy}{dx}=4x^3.$$

2. $y = 3x^4 + 4x^3.$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^4 + 4x^3) = \frac{d}{dx}(3x^4) + \frac{d}{dx}(4x^3),$$

by III., making $u = 3x^4$ and $v = 4x^3$.

$$\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4), \quad \text{by V.}$$

$$= 3 \cdot 4x^3 = 12x^3.$$

Similarly, $$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3) = 4 \cdot 3x^2 = 12x^2.$$

Hence, $$\frac{dy}{dx} = 12x^3 + 12x^2 = 12(x^3 + x^2).$$

3. $y = x^{\frac{3}{2}} + 2.$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{\frac{3}{2}}) + \frac{d}{dx}(2).$$

$$\frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}, \quad \text{by VII.}$$

$$\frac{d}{dx}(2) = 0, \quad \text{by II.}$$

Hence, $$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}.$$

4. $y = 3\sqrt{x} - \frac{2}{\sqrt{x}} + \frac{1}{x^3} + a.$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^{\frac{1}{2}}) - \frac{d}{dx}(2x^{-\frac{1}{2}}) + \frac{d}{dx}(x^{-3}) + \frac{da}{dx}$$

$$= \frac{3}{2}x^{-\frac{1}{2}} - 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} - 3x^{-4} + 0$$

$$= \frac{3}{2x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}} - \frac{3}{x^4}.$$

5. $y=\dfrac{x+3}{x^2+3}$.

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x+3}{x^2+3}\right).$$

Applying VI., making

$u=x+3$ and $v=x^2+3$, we have

$$\frac{d}{dx}\left(\frac{x+3}{x^2+3}\right)=\frac{(x^2+3)\frac{d}{dx}(x+3)-(x+3)\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$=\frac{x^2+3-(x+3)2x}{(x^2+3)^2}=\frac{3-6x-x^2}{(x^2+3)^2}.$$

Hence, $\dfrac{dy}{dx}=\dfrac{3-6x-x^2}{(x^2+3)^2}$.

6. $y=(x^2+2)^{\frac{2}{3}}$.

$$\frac{dy}{dx}=\frac{d}{dx}(x^2+2)^{\frac{2}{3}}.$$

If we apply VII., making

$u=x^2+2$ and $n=\dfrac{2}{3}$, we have

$$\frac{d}{dx}(x^2+2)^{\frac{2}{3}}=\frac{2}{3}(x^2+2)^{-\frac{1}{3}}\frac{d}{dx}(x^2+2)$$

$$=\frac{2}{3}(x^2+2)^{-\frac{1}{3}}2x=\frac{4x}{3(x^2+2)^{\frac{1}{3}}}.$$

Hence, $\dfrac{dy}{dx}=\dfrac{4x}{3(x^2+2)^{\frac{1}{3}}}$.

7. $y=(x^2+1)\sqrt{x^3-x}$.

$$\frac{dy}{dx}=\frac{d}{dx}[(x^2+1)(x^3-x)^{\frac{1}{2}}].$$

If we apply IV., making

$$u = x^2 + 1 \text{ and } v = (x^3 - x)^{\frac{1}{2}}, \text{ we have}$$

$$\frac{d}{dx}[(x^2+1)(x^3-x)^{\frac{1}{2}}]$$

$$= (x^2+1)\frac{d}{dx}(x^3-x)^{\frac{1}{2}} + (x^3-x)^{\frac{1}{2}}\frac{d}{dx}(x^2+1).$$

$$\frac{d}{dx}(x^3-x)^{\frac{1}{2}} = \frac{1}{2}(x^3-x)^{-\frac{1}{2}}\frac{d}{dx}(x^3-x) = \frac{1}{2}(x^3-x)^{-\frac{1}{2}}(3x^2-1).$$

$$\frac{d}{dx}(x^2+1) = 2x.$$

Hence $\frac{dy}{dx} = \frac{1}{2}(x^2+1)(3x^2-1)(x^3-x)^{-\frac{1}{2}} + (x^3-x)^{\frac{1}{2}}2x$

$$= \frac{(x^2+1)(3x^2-1) + 4x(x^3-x)}{2(x^3-x)^{\frac{1}{2}}} = \frac{7x^4-2x^2-1}{2(x^3-x)^{\frac{1}{2}}},$$

8. $y = 3x^{10} - 2x^6 + x^3 - 5,$ $\qquad \frac{dy}{dx} = 3(10x^9 - 4x^5 + x^2).$

9. $y = 6\sqrt[3]{x^2} + \frac{4}{\sqrt{x^3}} - \frac{2}{x} + \frac{3}{x^4},$ $\qquad \frac{dy}{dx} = \frac{4}{\sqrt[3]{x}} - \frac{6}{\sqrt{x^5}} + \frac{2}{x^2} - \frac{12}{x^5}.$

10. $y = (x+2a)(x-a)^2,$ $\qquad \frac{dy}{dx} = 3(x^2 - a^2).$

11. $y = (x^{\frac{1}{3}} - a^{\frac{1}{3}})^4,$ $\qquad \frac{dy}{dx} = \frac{4(x^{\frac{1}{3}} - a^{\frac{1}{3}})^3}{3x^{\frac{2}{3}}}.$

Differentiate Example 11 also after expanding.

12. $y = \frac{2x-1}{(x-1)^2},$ $\qquad \frac{dy}{dx} = -\frac{2x}{(x-1)^3}.$

13. $y = x(x^3+5)^{\frac{4}{3}}$ $\qquad \frac{dy}{dx} = 5(x^3+1)(x^3+5)^{\frac{1}{3}}.$

14. $y=\dfrac{x}{\sqrt{a^2-x^2}}$, $\qquad \dfrac{dy}{dx}=\dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$.

15. $y=\dfrac{(x^2-a^2)^{\frac{3}{2}}}{x^3}$, $\qquad \dfrac{dy}{dx}=\dfrac{3a^2\sqrt{x^2-a^2}}{x^4}$.

16. $y=\sqrt{\dfrac{a-x}{x}}$, $\qquad \dfrac{dy}{dx}=-\dfrac{a}{2x\sqrt{ax-x^2}}$.

Differentiate both members of the identical equations, Exs. 17–19

17. $(x^2+ax+a^2)(x^2-ax+a^2)=x^4+a^2x^2+a^4$.

18. $\left(\dfrac{x^2+a^2}{x}\right)^2=x^2+2a^2+a^4x^{-2}$.

19. $\dfrac{2x^2}{2x^2-3x+1}=1+\dfrac{2}{x-1}-\dfrac{1}{2x-1}$.

20. $x=t(t^2+a^2)^{\frac{n-1}{2}}$, $\qquad \dfrac{dx}{dt}=(nt^2+a^2)(t^2+a^2)^{\frac{n-3}{2}}$.

21. $y=\dfrac{(2t^2-3)^3}{(t^3+2)^2}$, $\qquad \dfrac{dy}{dt}=\dfrac{6(3t^2+4t)(2t^2-3)^2}{(t^3+2)^3}$.

22. $y=\dfrac{x^5(a+2x)^4}{(2a-3x)^9}$, $\qquad \dfrac{dy}{dx}=\dfrac{2ax^4(24x+5a)(a+2x)^3}{(2a-3x)^{10}}$.

23. $y=(x+1)^3(3x-8)^4(x+2)^6$,

$$\frac{dy}{dx}=3(13x^2-24)(x+1)^2(3x-8)^3(x+2)^5$$

24. $y=x(x^n+n)^{\frac{n-1}{n}}$, $\qquad \dfrac{dy}{dx}=\dfrac{n(x^n+1)}{(x^n+n)^{\frac{1}{n}}}$.

25. $y=\dfrac{x-a}{\sqrt{2ax-x^2}}$, $\qquad \dfrac{dy}{dx}=\dfrac{a^2}{(2ax-x^2)^{\frac{3}{2}}}$.

26. $y=\frac{(x-x^3)^{\frac{4}{3}}}{x^4}$, $\qquad \frac{dy}{dx}=-\frac{8\,(x-x^3)^{\frac{1}{3}}}{3\,x^4}$.

27. $y=\frac{2\,x^2+1}{x^3}\sqrt{1-x^2}$, $\qquad \frac{dy}{dx}=-\frac{3}{x^4\sqrt{1-x^2}}$.

28. $x=(t^{\frac{1}{2}}-2)\sqrt{t^{\frac{1}{2}}+1}$, $\qquad \frac{dx}{dt}=\frac{3}{4\,(t^{\frac{1}{2}}+1)^{\frac{1}{2}}}$.

29. $y=\frac{(x^3-a^3)^{\frac{1}{3}}}{(x^3+a^3)^{\frac{1}{3}}}$, $\qquad \frac{dy}{dx}=\frac{2\,a^3x^2}{(x^3+a^3)^{\frac{4}{3}}(x^3-a^3)^{\frac{2}{3}}}$.

30. $y=\sqrt{\frac{x^2-x+1}{x^2+x+1}}$, $\qquad \frac{dy}{dx}=\frac{x^2-1}{(x^2+x+1)\sqrt{x^4+x^2+1}}$.

31. $y=\frac{6\,x^2+6\,x+1}{(4\,x+1)^{\frac{3}{2}}}$, $\qquad \frac{dy}{dx}=\frac{12\,x^2}{(4\,x+1)^{\frac{5}{2}}}$.

32. $y=(x^2-3\,ax)^{\frac{4}{5}}(4x^2+8\,ax+15\,a^2)^{\frac{1}{5}}$,

$$\frac{dy}{dx}=\frac{4\,(2\,x^3-9\,a^3)}{(x^2-3\,ax)^{\frac{1}{5}}(4\,x^2+8\,ax+15\,a^2)^{\frac{4}{5}}}.$$

33. $y=(x+\sqrt{x^2+1})^n(n\sqrt{x^2+1}-x)$,

$$\frac{dy}{dx}=(n^2-1)(x+\sqrt{x^2+1})^n.$$

34. For what values of x is $3\,x^4-8\,x^3$ an increasing or a decreasing function of x?

Ans. Increasing, when $x>2$; decreasing, when $x<2$

35. A vessel in the form of an inverted circular cone of semi-vertical angle 30°, is being filled with water at the uniform rate of one cubic foot per minute. At what rate is the surface of the water rising when the depth is 6 inches? when 1 foot? when 2 feet?

Ans. .76 in.; .19 in.; .05 in., per sec.

36. The side of an equilateral triangle is increasing at the rate of 10 feet per minute, and the area at the rate of 10 square feet per second. How large is the triangle? *Ans.* Side = 69.28 ft.

37. A vessel is sailing due north 20 miles per hour. Another vessel, 40 miles north of the first, is sailing due east 15 miles per hour. At what rate are they approaching each other after one hour? After 2 hours? *Ans.* Approaching 7 mi. per hr.; separating 15 mi. per hr.

When will they cease to approach each other, and what is then their distance apart?

Ans. After 1 hr. 16 min. 48 sec. Distance = 24 mi.

38. A train starts at noon from Boston, moving west, its motion being represented by $s = 9t^2$. From Worcester, forty miles west of Boston, another train starts at the same time, moving in the same direction, its motion represented by $s' = 2t^3$. The quantities s, s', are in miles, and t in hours. When will the trains be nearest together, and what is then their distance apart?

Ans. 3 P.M., and 13 mi.

When will the accelerations be equal?

Ans. 1 hr. 30 min., P.M.

39. If a point moves so that $s = \sqrt{t}$, show that the acceleration is negative and proportional to the cube of the velocity. How is the sign of the acceleration interpreted?

40. Given $s = \frac{a}{t} + bt^2$; find the velocity and acceleration.

41. A body starts from the origin, and moves so that in t seconds the coördinates of its position are

$$x = t^3 + 4t^2 - 3t, \qquad y = \frac{4t^3}{3} - 3t^2 - 4t.$$

Find the rates of increase of x and y.

Also find the velocity in its path, which is

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Ans. $5t^2 + 5$.

42. Two bodies move, one on the axis of X, and the other on the axis of Y, and in t minutes their distances from the origin are

$$x = 2t^2 - 6t \text{ feet, and } y = 6t - 9 \text{ feet.}$$

At what rate are they approaching each other or separating, after 1 minute? After 3 minutes?

Ans. Approaching 2 ft. per min.; separating 6 ft. per min.

When will they be nearest together? *Ans.* After 1 min. 30 sec.

43. In the triangle ABC, L and M are the middle points of BC and CA respectively. A man walks along the median AL at a uniform rate. A lamp at B casts his shadow on the side AC. Show that the velocities of the shadow at A, M, C, are as $2^2 : 3^2 : 4^2$; and that the accelerations at these points are as $2^3 : 3^3 : 4^3$.

SUGGESTION. — P being any position of the man, draw from L a line parallel to BP.

33. Formulæ for Differentiation of Logarithmic and Exponential Functions.

$$\text{VIII.}\quad \frac{d}{dx}\log_a u = \log_a e \frac{\frac{du}{dx}}{u}.$$

$$\text{IX.}\quad \frac{d}{dx}\log_e u = \frac{\frac{du}{dx}}{u}.$$

$$\text{X.}\quad \frac{d}{dx}a^u = \log_e a \cdot a^u \frac{du}{dx}.$$

$$\text{XI.}\quad \frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

$$\text{XII.}\quad \frac{d}{dx}u^v = vu^{v-1}\frac{du}{dx} + \log_e u \cdot u^v \frac{dv}{dx}.$$

34. Proof of VIII. Let $y = \log_a u$,

then
$$y + \Delta y = \log_a (u + \Delta u),$$

$$\Delta y = \log_a (u + \Delta u) - \log_a u = \log_a \frac{u + \Delta u}{u}$$

$$= \log_a\left(1 + \frac{\Delta u}{u}\right) = \frac{\Delta u}{u} \log_a \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \log_a \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} \frac{\frac{\Delta u}{\Delta x}}{u}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If Δx approach zero, Δu likewise approaches zero.

Now $\text{Lim}_{\Delta u = 0} \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} = \text{Lim}_{z = \infty} \left(1 + \frac{1}{z}\right)^z.$

For, if we put
$$\frac{u}{\Delta u} = z,$$

then
$$\left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} = \left(1 + \frac{1}{z}\right)^z,$$

and as Δu approaches zero, z approaches infinity.

But in Art. 12 we have found

$$\text{Lim}_{z = \infty} \left(1 + \frac{1}{z}\right)^z = e;$$

therefore $\text{Lim}_{\Delta u = 0} \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} = e.$

Hence, if we take the limit of each member of (1),

$$\frac{dy}{dx} = \log_a e \frac{\frac{du}{dx}}{u}.$$

35. Proof of IX. This is a special case of VIII., when $a = e$. In this case

$$\log_a e = \log_e e = 1.$$

NOTE.—Logarithms to base e are called *Napierian* logarithms. Hereafter, when no base is specified, Napierian logarithms are to be understood; that is,

$$\log u \text{ denotes } \log_e u.$$

36. Proof of X.

Let
$$y = a^u.$$

Taking the logarithm of each member, we have

$$\log y = u \log a;$$

therefore by IX.,
$$\frac{\frac{dy}{dx}}{y} = \log a \frac{du}{dx}.$$

Multiplying by $y = a^u$, we have

$$\frac{dy}{dx} = \log a \cdot a^u \frac{du}{dx}.$$

37. Proof of XI. This is a special case of X., where $a = e$.

38. Proof of XII. Let $y = u^v$.

Taking the logarithm of each member, we have

$$\log y = v \log u;$$

therefore by IX.,
$$\frac{\frac{dy}{dx}}{y} = \frac{v\frac{du}{dx}}{u} + \log u \frac{dv}{dx}.$$

Multiplying by $y = u^v$, we have

$$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + \log u \cdot u^v \frac{dv}{dx}.$$

The method of proving X. and XII. by taking the logarithm of each member, may be applied to IV., VI., and VII.

This exercise is left to the student.

EXAMPLES

(See note, Art. 85.)

1. $y = \log(2x^3 + 3x^2)$, $\qquad \dfrac{dy}{dx} = \dfrac{6(x+1)}{2x^2 + 3x}$.

2. $y = x^n \log(ax + b)$, $\qquad \dfrac{dy}{dx} = x^{n-1}\left[\dfrac{ax}{ax+b} + n\log(ax+b)\right]$

3. $y = \dfrac{1}{x \log x}$, $\qquad \dfrac{dy}{dx} = -\dfrac{1 + \log x}{(x\log x)^2}$.

4. $y = \log_{10}(3x + 2)$, $\qquad \dfrac{dy}{dx} = \dfrac{3\log_{10} e}{3x + 2} = \dfrac{1.3029}{3x+2}$.

5. $y = \log\dfrac{ax - b}{ax + b}$, $\qquad \dfrac{dy}{dx} = \dfrac{2ab}{a^2x^2 - b^2}$.

6. $y = \log\dfrac{3x+1}{x+3}$, $\qquad \dfrac{dy}{dx} = \dfrac{8}{3x^2 + 10x + 3}$.

7. $y = \log\dfrac{t^2 - t + 1}{t^2 + t + 1}$, $\qquad \dfrac{dy}{dt} = \dfrac{2(t^2 - 1)}{t^4 + t^2 + 1}$.

8. $y = a^x e^x$, $\qquad \dfrac{dy}{dx} = (1 + \log a)a^x e^x$.

9. $y = \log(a^x + b^x)$, $\qquad \dfrac{dy}{dx} = \dfrac{a^x \log a + b^x \log b}{a^x + b^x}$.

10. $y = (e^{2x} - 1)^4$, $\qquad \dfrac{dy}{dx} = 8e^{2x}(e^{2x} - 1)^3$.

Differentiate Ex. 10 also after expanding.

11. $y = \dfrac{5x+3}{x-3}e^{-2x}$, $\qquad \dfrac{dy}{dx} = \dfrac{24x - 10x^2}{(x-3)^2}e^{-2x}$.

12. $y = (3x - 1)^2 e^{3x-2}$, $\qquad \dfrac{dy}{dx} = 3(9x^2 - 1)e^{3x-2}$.

13. $y = x^5 5^x$, $\qquad \dfrac{dy}{dx} = x^4 5^x(5 + x \log 5)$.

14. $y = \log \log x - \dfrac{1}{\log x}$, $\qquad \dfrac{dy}{dx} = \dfrac{1 + \log x}{x(\log x)^2}$.

Differentiate both members of the identical equations, **Exs. 15–18.**

15. $(x + e^x)^4 = x^4 + 4\,x^3e^x + 6\,x^2e^{2x} + 4\,xe^{3x} + e^{4x}$.

16. $(a^x - e^x)^3 = a^{3x} - 3\,a^{2x}e^x + 3\,a^xe^{2x} - e^{3x}$.

17. $\log(e^{2x} + e^{2a}) = \log(e^{x-a} + e^{a-x}) + x + a$.

18. $x^{\log a} = a^{\log x}$.

19. $y = \dfrac{x \log x}{x+1} - \log(x+1)$, $\qquad \dfrac{dy}{dx} = \dfrac{\log x}{(x+1)^2}$.

20. $y = \log(\sqrt{x + a} + \sqrt{x})$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x^2 + ax}}$.

21. $y = \log(2\,x + \sqrt{4\,x^2 - 1})$, $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sqrt{4\,x^2 - 1}}$.

22. $y = \log \dfrac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x+1}}$.

23. $y = x[(\log x)^2 - 2\log x + 2]$, $\qquad \dfrac{dy}{dx} = (\log x)^2$.

24. $y = \log(\sqrt[3]{x+a} - \sqrt[3]{x-a})$, $\qquad \dfrac{dy}{dx} = -\dfrac{(x+a)^{\frac{1}{3}} + (x-a)^{\frac{1}{3}}}{3(x^2 - a^2)^{\frac{2}{3}}}$.

25. $y = \log(\sqrt{x+3} + \sqrt{x+2}) + \sqrt{(x+3)(x+2)}$, $\qquad \dfrac{dy}{dx} = \sqrt{\dfrac{x+3}{x+2}}$.

26. $y = \log \dfrac{e^{nx} - 1 + e^{-nx}}{e^{nx} + 1 + e^{-nx}}$, $\qquad \dfrac{dy}{dx} = \dfrac{2\,n(e^{nx} - e^{-nx})}{e^{2nx} + 1 + e^{-2nx}}$.

27. $y=\log\frac{\log x}{1+\log x}$, $\frac{dy}{dx}=\frac{1}{x\log x\,(1+\log x)}$.

28. $y=(3e^{4x}-2e^{2x}+2)\sqrt{2e^{2x}+1}$, $\frac{dy}{dx}=\frac{30e^{6x}}{\sqrt{2e^{2x}+1}}$.

29. $y=3\log(\sqrt{x^2+3}-3)+\log(\sqrt{x^2+3}+1)$,

$$\frac{dy}{dx}=\frac{4x}{x^2-2\sqrt{x^2+3}}.$$

30. $y=\log(a+\sqrt{2ax-a^2})+\frac{\sqrt{a}}{\sqrt{a}+\sqrt{2x-a}}$,

$$\frac{dy}{dx}=\frac{1}{2(x+\sqrt{2ax-a^2})}.$$

31. $y=\log\frac{x^2+1+\sqrt{x^4+3x^2+1}}{x}$, $\frac{dy}{dx}=\frac{x^2-1}{x\sqrt{x^4+3x^2+1}}$.

32. $y=\log_x(x+a)$, $\frac{dy}{dx}=\frac{1}{\log x}\left[\frac{1}{x+a}-\frac{\log_x(x+a)}{x}\right]$.

The following may be derived by XII. or by differentiating after taking the logarithm of each member of the given equation.

33. $y=x^{nx}$, $\frac{dy}{dx}=nx^{nx}(1+\log x)$.

34. $y=(ax^2)^x$, $\frac{dy}{dx}=(ax^2)^x[2+\log(ax^2)]$.

35. $y=x^{ax^2}$, $\frac{dy}{dx}=ax^{ax^2+1}(1+2\log x)$.

36. $y=(\log x)^x$, $\frac{dy}{dx}=(\log x)^x\left(\frac{1}{\log x}+\log\log x\right)$.

37. $y=x^{(\log x)^n}$, $\frac{dy}{dx}=(n+1)(\log x)^n x^{(\log x)^n-1}$.

38. $y=\left(\frac{x}{x+a}\right)^{\frac{x}{a}}$, $\frac{dy}{dx}=\left(\frac{x}{x+a}\right)^{\frac{x}{a}}\left(\frac{1}{x+a}+\frac{1}{a}\log\frac{x}{x+a}\right)$.

The method of differentiating after taking the logarithm of the expression may often be applied with advantage to algebraic functions. This is sometimes called *logarithmic differentiation.*

In this way differentiate Exs. 21–26, pp. 36, 37.

39. Find the slope of the catenary (Art. 128) $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$, at $x=0$.

What is the abscissa of the point where the curve is inclined 45° to the axis of X? *Ans.* $x=a\log_e(1+\sqrt{2})$.

40. When does $\log_{10}x$ increase at the same rate as x?

Ans. When $x=\log_{10}e=.4343$.

When at one third the rate? *Ans.* When $x=1.3029$.

Verify these results from logarithm tables.

41. If the space described by a point is given by $s=ae^t+be^{-t}$, show that the acceleration is equal to the space passed over.

42. If a point moves so that in t seconds $s=10\log\frac{4}{t+4}$ feet, find the velocity and acceleration at the end of 1 second. At the end of 16 seconds. *Ans.* Velocity $=-2$ ft., and $-.5$ ft. per sec.

Acceleration $=.4$, and $.025$.

43. For what values of x is $y=\log(x-2)^3-\frac{9x^2-36x+32}{(x-2)^3}$ an increasing or a decreasing function?

Ans. Increasing when $x>3$; decreasing when $x<3$.

39. Formulæ for Differentiation of Trigonometric Functions. In the following formulæ the angle u is supposed to be expressed in circular measure.

$$\text{XIII.}\quad \frac{d}{dx}\sin u=\cos u\frac{du}{dx}.$$

$$\text{XIV.}\quad \frac{d}{dx}\cos u=-\sin u\frac{du}{dx}.$$

$$\text{XV.}\quad \frac{d}{dx}\tan u=\sec^2 u\frac{du}{dx}.$$

XVI. $\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u\frac{du}{dx}.$

XVII. $\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}.$

XVIII. $\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u \cot u\frac{du}{dx}.$

XIX. $\frac{d}{dx}\operatorname{vers} u = \sin u\frac{du}{dx}.$

40. Proof of XIII. Let $y = \sin u$,

then $$y + \Delta y = \sin(u + \Delta u);$$

therefore $$\Delta y = \sin(u + \Delta u) - \sin u.$$

But from Trigonometry,

$$\sin A - \sin B = 2\sin\frac{1}{2}(A - B)\cos\frac{1}{2}(A + B).$$

If we substitute $A = u + \Delta u$, and $B = u$,

we have $$\Delta y = 2\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2}.$$

Hence, $$\frac{\Delta y}{\Delta x} = \cos\left(u + \frac{\Delta u}{2}\right)\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}\frac{\Delta u}{\Delta x}.$$

Now when Δx approaches zero, Δu likewise approaches zero, and as Δu is in circular measure,

$$\operatorname{Lim}_{\Delta u=0}\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}} = 1. \quad \text{See Art. 12.}$$

Hence, $$\frac{dy}{dx} = \cos u\frac{du}{dx}.$$

41. Proof of XIV. This may be derived by substituting in XIII for u, $\frac{\pi}{2}-u$.

Then $$\frac{d}{dx}\sin\left(\frac{\pi}{2}-u\right)=\cos\left(\frac{\pi}{2}-u\right)\frac{d}{dx}\left(\frac{\pi}{2}-u\right),$$

or $$\frac{d}{dx}\cos u=\sin u\left(-\frac{du}{dx}\right)=-\sin u\,\frac{du}{dx}.$$

42. Proof of XV. Since $\tan u=\frac{\sin u}{\cos u}$,

by VI., $$\frac{d}{dx}\tan u=\frac{\cos u\frac{d}{dx}\sin u-\sin u\frac{d}{dx}\cos u}{\cos^2 u}$$

$$=\frac{\cos^2 u\frac{du}{dx}+\sin^2 u\frac{du}{dx}}{\cos^2 u}=\frac{\frac{du}{dx}}{\cos^2 u}$$

$$=\sec^2 u\,\frac{du}{dx}.$$

43. Proof of XVI. This may be derived from XV. by substituting $\frac{\pi}{2}-u$ for u.

44. Proof of XVII. Since $\sec u=\frac{1}{\cos u}$,

by VI., $$\frac{d}{dx}\sec u=\frac{-\frac{d}{dx}\cos u}{\cos^2 u}=\frac{\sin u\frac{du}{dx}}{\cos^2 u}$$

$$=\sec u\tan u\frac{du}{dx}.$$

45. Proof of XVIII. This may be derived from XVII. by substituting $\frac{\pi}{2}-u$ for u.

46. Proof of XIX. This is readily obtained from XIV. by the relation

$$\text{vers}\,u=1-\cos u.$$

EXAMPLES

1. $y = 3 \sin 3x \cos 2x - 2 \cos 3x \sin 2x, \quad \frac{dy}{dx} = 5 \cos 3x \cos 2x.$

2. $y = \log \cos^2 x + 2x \tan x - x^2, \quad \frac{dy}{dx} = 2x \tan^2 x.$

3. $y = \log (\sec mx + \tan mx), \quad \frac{dy}{dx} = m \sec mx.$

4. $y = \log (a \sin^2 x + b \cos^2 x), \quad \frac{dy}{dx} = \frac{2(a-b) \tan x}{a \tan^2 x + b}.$

5. $y = \cos \alpha \log \sec (\theta - \alpha) + \theta \sin \alpha, \quad \frac{dy}{d\theta} = \frac{\sin \theta}{\cos (\theta - \alpha)}.$

6. $y = (m-1) \sec^{m+1} x - (m+1) \sec^{m-1} x, \quad \frac{dy}{dx} = (m^2 - 1) \sec^{m-1} x \tan^3 x.$

7. $y = \log \tan \left(ax - \frac{\pi}{4}\right), \quad \frac{dy}{dx} = -2a \sec 2ax.$

8. $r = \log [\sec \theta \tan \theta (\sec \theta + \tan \theta)^2], \quad \frac{dr}{d\theta} = \frac{(\sec \theta + \tan \theta)^2}{\tan \theta}.$

9. $y = \operatorname{cosec}^m ax \operatorname{cosec}^n bx,$

$$\frac{dy}{dx} = -\operatorname{cosec}^m ax \operatorname{cosec}^n bx \,(ma \cot ax + nb \cot bx).$$

10. $y = 2x^2 \sin 2x + 2x \cos 2x - \sin 2x, \quad \frac{dy}{dx} = 4x^2 \cos 2x.$

11. $y = 2 \tan^3 x \sec x + \tan x \sec x - \log (\sec x + \tan x),$

$$\frac{dy}{dx} = 8 \tan^2 x \sec^3 x.$$

12. $y = \frac{\sin x + \cos x}{e^x}, \quad \frac{dy}{dx} = -\frac{2 \sin x}{e^x}.$

13. $y = e^{3x}(\sin 2x - 5 \cos 2x), \quad \frac{dy}{dx} = 13e^{3x}(\sin 2x - \cos 2x).$

14. $y = \log \dfrac{\cos x}{\cos (x+a)}$, $\qquad \dfrac{dy}{dx} = \dfrac{\sin a}{\cos x \cos (x+a)}$.

15. $y = \sin^3 4x \cos^4 3x$, $\qquad \dfrac{dy}{dx} = 12 \sin^2 4x \cos^3 3x . \cos 7x$.

16. $y = \log \dfrac{\sin x + \operatorname{vers} x}{\sin x - \operatorname{vers} x}$, $\qquad \dfrac{dy}{dx} = \sec x$.

17. $y = (\sin 2x)^x$, $\qquad \dfrac{dy}{dx} = y\,(\log \sin 2x + 2x \cot 2x)$.

18. $y = (\tan x)^{\sin x}$, $\qquad \dfrac{dy}{dx} = y\,(\cos x \log \tan x + \sec x)$.

19. $y = (\sin x)^{\log \cos x}$, $\qquad \dfrac{dy}{dx} = y\,(\cot x \log \cos x - \tan x \log \sin x)$.

20. $y = \tan x \sec x + \log \sqrt{\dfrac{1+\sin x}{1-\sin x}}$, $\qquad \dfrac{dy}{dx} = 2 \sec^3 x$.

21. $y = (\tan x - 3 \cot x) \sqrt{\tan x}$, $\qquad \dfrac{dy}{dx} = \dfrac{3 \sec^4 x}{2 \tan^{\frac{3}{2}} x}$.

22. $y = \log \dfrac{\sin \frac{1}{2}(\theta - \alpha)}{\sin \frac{1}{2}(\theta + \alpha)}$, $\qquad \dfrac{dy}{d\theta} = \dfrac{\sin \alpha}{\cos \alpha - \cos \theta}$.

23. $y = a \log (a \sin x + b \cos x) + bx$, $\qquad \dfrac{dy}{dx} = \dfrac{a^2 + b^2}{a \tan x + b}$.

24. $y = \dfrac{\sin \left(2x - \frac{\pi}{4}\right)}{\sin \left(2x + \frac{\pi}{4}\right)}$, $\qquad \dfrac{dy}{dx} = \dfrac{4}{1 + \sin 4x}$.

25. $y = \log \dfrac{\tan \frac{x}{2} - 2}{2 \tan \frac{x}{2} - 1}$, $\qquad \dfrac{dy}{dx} = \dfrac{3}{4 - 5 \sin x}$.

26. $y=\frac{a \sin x+b \operatorname{vers} x}{a \sin x-b \operatorname{vers} x}$, $\qquad \frac{dy}{dx}=\frac{2ab \operatorname{vers} x}{(a \sin x-b \operatorname{vers} x)^2}$.

In each of the following pairs of equations derive by differentiation each of the two equations from the other:

27. $\sin 2x=2 \sin x \cos x,$
$\cos 2x=\cos^2 x-\sin^2 x.$

28. $\sin 2x=\frac{2 \tan x}{1+\tan^2 x},$
$\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}.$

29. $\sin 3x=3 \sin x-4 \sin^3 x,$
$\cos 3x=4 \cos^3 x-3 \cos x.$

30. $\sin 4x=4 \sin x \cos^3 x-4 \cos x \sin^3 x,$
$\cos 4x=1-8 \sin^2 x \cos^2 x.$

31. $\sin (m+n)x=\sin mx \cos nx+\cos mx \sin nx,$
$\cos (m+n)x=\cos mx \cos nx-\sin mx \sin nx.$

32. If θ vary uniformly, so that one revolution is made in π seconds, show that the rates of increase of $\sin \theta$, when $\theta = 0°, 30°, 45°, 60°, 90°$, are respectively $2, \sqrt{3}, \sqrt{2}, 1, 0$, per second.

33. If θ is increasing uniformly, show that the rates of increase of $\tan \theta$, when $\theta=0°, 30°, 45°, 60°, 90°$, are in harmonical progression.

34. For what values of θ, less than 90°, is $\sin \theta+\cos \theta$ an increasing or a decreasing function? Find its rate of change when $\theta=15°$, the rate of change of θ being ω. *Ans.* $\frac{\omega}{\sqrt{2}}$.

35. The crank and connecting rod of a steam engine are 3 and 10 feet respectively, and the crank revolves uniformly, making two revolutions per second. At what rate is the piston moving, when the crank makes with the line of motion of the piston 0°, 45°, 90°, 135°, 180°?

If a, b, x, are the three sides of the triangle, and θ the angle opposite b,

$$x = a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta}.$$

Ans. 0, 32.44, 37.70, 20.87, 0, ft. per sec.

36. A crank OP revolves about O with angular velocity ω, and a connecting rod PQ is hinged to it at P, whilst Q is constrained to move in a fixed groove OX. Prove that the velocity of Q is $\omega . OR$, where R is the point in which the line QP (produced if necessary) meets a perpendicular to OX drawn through O.

47. Inverse Trigonometric Functions. The inverse trigonometric functions are many-valued functions; that is, for any given value of x, there is an infinite number of values of $\sin^{-1}x$, $\tan^{-1}x$, &c.

For example, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6} \pm 2n\pi$ or $\frac{5\pi}{6} \pm 2n\pi$, where n is any integer.

But if the angle is restricted to values not greater numerically than a right angle, $\sin^{-1}x$ will have only *one* value for a given value of x. Then $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. We thus regard $\sin^{-1}x$, $\operatorname{cosec}^{-1}x$, $\tan^{-1}x$, and $\cot^{-1}x$, as taken between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, that is, in the first or fourth quadrants.

But $\cos^{-1}x$, $\sec^{-1}x$, and $\operatorname{vers}^{-1}x$, must be taken between 0 and π, that is, in the first and second quadrants, which include all values of the cosine, secant, and versine.

These restrictions are assumed in the following formulæ of differentiation.

48. Formulæ for Differentiation of Inverse Trigonometric Functions.

$$\text{XX.}\quad \frac{d}{dx}\sin^{-1}u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

$$\text{XXI.}\quad \frac{d}{dx}\cos^{-1}u = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

$$\text{XXII.}\quad \frac{d}{dx}\tan^{-1}u = \frac{\frac{du}{dx}}{1+u^2}.$$

$$\text{XXIII.}\quad \frac{d}{dx}\cot^{-1}u = -\frac{\frac{du}{dx}}{1+u^2}.$$

$$\text{XXIV.}\quad \frac{d}{dx}\sec^{-1}u = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$\text{XXV.}\quad \frac{d}{dx}\operatorname{cosec}^{-1}u = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$\text{XXVI.}\quad \frac{d}{dx}\operatorname{vers}^{-1}u = \frac{\frac{du}{dx}}{\sqrt{2u-u^2}}.$$

49. Proof of XX. Let $y = \sin^{-1}u$;

therefore $$\sin y = u.$$

By XIII., $$\cos y\frac{dy}{dx} = \frac{du}{dx};$$

therefore $$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\cos y}.$$

But $$\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-u^2}.$$

If the angle y is restricted to the first and fourth quadrants (Art. 47), $\cos y$ is positive.

Hence $$\cos y = \sqrt{1-u^2},$$

and $$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

50. Proof of XXI. Let $y = \cos^{-1} u$;

therefore $$\cos y = u.$$

By XIV., $$-\sin y \frac{dy}{dx} = \frac{du}{dx};$$

therefore $$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\sin y}.$$

But $$\sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-u^2}.$$

If the angle y is restricted to the first and second quadrants (Art. 47), $\sin y$ is positive.

Hence $$\sin y = \sqrt{1-u^2},$$

and $$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

51. Proof of XXII. Let $y = \tan^{-1} u$;

therefore $$\tan y = u.$$

By XV., $$\sec^2 y \frac{dy}{dx} = \frac{du}{dx};$$

therefore $$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\sec^2 y}.$$

But $$\sec^2 y = 1 + \tan^2 y = 1 + u^2;$$

therefore $$\frac{dy}{dx} = \frac{\frac{du}{dx}}{1+u^2}.$$

52. Proof of XXIII. This may be derived like XXII., or from $\cot^{-1} u = \tan^{-1}\frac{1}{u}$.

53. Proof of XXIV. This may be obtained from XXI. Since $\sec^{-1} u = \cos^{-1}\frac{1}{u}$,

$$\frac{d}{dx}\sec^{-1} u = \frac{d}{dx}\cos^{-1}\frac{1}{u} = -\frac{\frac{d}{dx}\left(\frac{1}{u}\right)}{\sqrt{1-\frac{1}{u^2}}} = -\frac{-\frac{1}{u^2}\frac{du}{dx}}{\sqrt{1-\frac{1}{u^2}}} = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$$

54. Proof of XXV. This may be obtained from XX. Since $\operatorname{cosec}^{-1} u = \sin^{-1}\frac{1}{u}$,

$$\frac{d}{dx}\operatorname{cosec}^{-1} u = \frac{d}{dx}\sin^{-1}\frac{1}{u} = \frac{\frac{d}{dx}\left(\frac{1}{u}\right)}{\sqrt{1-\frac{1}{u^2}}} = \frac{-\frac{1}{u^2}\frac{du}{dx}}{\sqrt{1-\frac{1}{u^2}}} = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$$

55. Proof of XXVI. This may be obtained from XXI. Since $\operatorname{vers}^{-1} u = \cos^{-1}(1-u)$,

$$\frac{d}{dx}\operatorname{vers}^{-1} u = \frac{d}{dx}\cos^{-1}(1-u) = -\frac{\frac{d}{dx}(1-u)}{\sqrt{1-(1-u)^2}} = \frac{\frac{du}{dx}}{\sqrt{2u-u^2}}.$$

EXAMPLES

1. $y = \tan^{-1}\frac{5x-1}{2}$, $\frac{dy}{dx} = \frac{2}{5x^2-2x+1}$.

2. $y = \sec^{-1}\frac{2x}{3}$, $\frac{dy}{dx} = \frac{3}{x\sqrt{4x^2-9}}$.

3. $y = \sin^{-1}\frac{2x-7}{3}$, $\frac{dy}{dx} = \frac{1}{\sqrt{(x-5)(2-x)}}$.

4. $y = \operatorname{vers}^{-1}(8x^2-8x^4)$, $\frac{dy}{dx} = \frac{4}{\sqrt{1-x^2}}$.

5. $y = \tan^{-1}\frac{x-a}{x+a}$, $\frac{dy}{dx} = \frac{a}{x^2+a^2}$.

6. $y = \tan^{-1}(3\tan\theta)$, $\qquad \frac{dy}{d\theta} = \frac{3}{5 - 4\cos 2\theta}$.

7. $y = \sec^{-1}\sec^2\theta$, $\qquad \frac{dy}{d\theta} = \frac{2}{\sqrt{\sec^2\theta + 1}}$.

8. $y = \operatorname{vers}^{-1}\frac{2}{x^2+1}$, $\qquad \frac{dy}{dx} = -\frac{2}{x^2+1}$.

9. $y = \cot^{-1}\frac{e^{ax} + e^{-ax}}{e^{ax} - e^{-ax}}$, $\qquad \frac{dy}{dx} = \frac{2a}{e^{2ax} + e^{-2ax}}$.

10. $y = \operatorname{cosec}^{-1}\frac{3x}{x-1}$, $\qquad \frac{dy}{dx} = \frac{1}{x\sqrt{8x^2 + 2x - 1}}$.

11. $y = \tan^{-1}\frac{3x-2}{5} + \cot^{-1}\frac{3x-12}{6x+1}$, $\qquad \frac{dy}{dx} = 0$.

12. $y = \cos^{-1}\sqrt{\operatorname{vers} x}$, $\qquad \frac{dy}{dx} = -\frac{1}{2}\sqrt{1 + \sec x}$.

13. $y = a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}$, $\qquad \frac{dy}{dx} = \frac{(a^2 - b^2)x^2}{(x^2 + a^2)(x^2 + b^2)}$.

14. $y = \cot^{-1}\frac{x + ab}{bx - a}$, $\qquad \frac{dy}{dx} = \frac{a}{x^2 + a^2}$.

15. $y = \sin^{-1}\frac{m\sin x - n\cos x}{\sqrt{m^2 + n^2}}$, $\qquad \frac{dy}{dx} = 1$.

16. $y = \sin^{-1}\frac{ax + b}{bx + a}$, $\qquad \frac{dy}{dx} = \frac{1}{bx + a}\sqrt{\frac{a^2 - b^2}{1 - x^2}}$.

17. $y = \tan^{-1}(\sec x + \tan x)$, $\qquad \frac{dy}{dx} = \frac{1}{2}$.

18. $y = \sin^{-1}\frac{2}{e^x + e^{-x}}$, $\qquad \frac{dy}{dx} = -\frac{2}{e^x + e^{-x}}$.

19. $y = \cot^{-1}(x^2 - x + 1) - \cot^{-1}(x - 1)$, $\qquad \frac{dy}{dx} = \frac{1}{x^2 + 1}$.

20. $y = \tan^{-1}\dfrac{4 + 5\tan x}{3}$, $\qquad \dfrac{dy}{dx} = \dfrac{3}{5 + 4\sin 2x}$.

21. $y = \cos^{-1}\dfrac{x-3}{3} - 2\sqrt{\dfrac{6-x}{x}}$, $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{6x - x^2}}{x^2}$.

22. $y = x^2 \sec^{-1}\dfrac{x}{2} - 2\sqrt{x^2 - 4}$, $\qquad \dfrac{dy}{dx} = 2x\sec^{-1}\dfrac{x}{2}$.

Differentiate both members of the identical equations, Exs. 23–28.

23. $2\cos^{-1}\sqrt{\dfrac{x+1}{2}} = \cos^{-1} x$.

24. $3\,\text{vers}^{-1} x = \text{vers}^{-1}\left[x(2x-3)^2\right]$.

25. $\sin^{-1} x + \sin^{-1} a = \sin^{-1}(a\sqrt{1-x^2} + x\sqrt{1-a^2})$.

26. $\tan^{-1} mx + \tan^{-1} nx = \tan^{-1}\dfrac{(m+n)x}{1 - mnx^2}$.

27. $\text{vers}^{-1}\dfrac{2x+2}{x+3} = 2\tan^{-1}\sqrt{\dfrac{x+1}{2}}$.

28. $\tan^{-1}\dfrac{a^2\tan x - b^2}{ab(1 + \tan x)} = \tan^{-1}\left(\dfrac{a}{b}\tan x\right) - \tan^{-1}\dfrac{b}{a}$.

29. $y = 2\log\dfrac{x^2 - 2x + 5}{x^2 + 2x + 5} + \tan^{-1}\dfrac{x^2 - 5}{4x}$, $\qquad \dfrac{dy}{dx} = \dfrac{12x^2 - 20}{x^4 + 6x^2 + 25}$.

30. $y = \tan^{-1}\dfrac{4(x - x^3)}{1 - 6x^2 + x^4}$, $\qquad \dfrac{dy}{dx} = \dfrac{4}{1 + x^2}$.

31. What value must be assigned to a so that the curve

$$y = \log_e (x - 7a) + \tan^{-1} ax,$$

may be parallel to the axis of X at the point $x = 1$?

Ans. $\frac{1}{2}$ or $-\frac{1}{3}$.

32. A man walks across the diameter, 200 feet, of a circular courtyard at a uniform rate of 5 feet per second. A lamp at one extremity of a diameter perpendicular to the first casts his shadow upon the circular wall. Required the velocity of the shadow along the wall, when he is at the centre; when 20 feet from the centre; when 50 feet; when 75 feet; when at circumference.

Ans. 10, $9\frac{8}{13}$, 8, $6\frac{2}{5}$, 5 ft. per sec.

33. Suppose the lamp in Ex. 32 to be moved halfway towards the centre; find the position of the man, when the velocity of the shadow equals his own.

Ans. Distance from centre $\frac{100\sqrt{15}}{6} = 64.55$ ft.

56. Relations between Certain Derivatives. It is necessary to notice the relations between certain derivatives obtained by differentiating with respect to different quantities.

To express $\frac{dy}{dx}$ *in terms of* $\frac{dx}{dy}$. If y is a given function of x, then x may be regarded as a function of y. From the former relation, we have $\frac{dy}{dx}$, and from the latter, $\frac{dx}{dy}$. These derivatives are connected by a simple relation.

It is evident that
$$\frac{\Delta y}{\Delta x} = \frac{1}{\dfrac{\Delta x}{\Delta y}},$$
however small the values of Δx and Δy. As these quantities approach zero, we have for the limits of the members of this equation,
$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}. \qquad (1)$$
That is, the relation between $\frac{dy}{dx}$ and $\frac{dx}{dy}$ is the same as if they were ordinary fractions.

For example, suppose
$$x = \frac{a}{y+1}. \qquad (2)$$

Differentiating with respect to y, we have
$$\frac{dx}{dy} = -\frac{a}{(y+1)^2}.$$

By (1),
$$\frac{dy}{dx} = -\frac{(y+1)^2}{a} = -\frac{a}{x^2}, \quad \text{by (2).}$$

This is the same result as that obtained by solving (2) with reference to y, giving

$$y=\frac{a}{x}-1,$$

and differentiating this with respect to x.

To express $\frac{dy}{dx}$ *in terms of* $\frac{dy}{dz}$ and $\frac{dz}{dx}$; that is, *to find the derivative of a function of a function.* If y is a given function of z, and z a given function of x, it follows that y is a function of x. This relation may often be obtained by eliminating z between the two given equations, but $\frac{dy}{dx}$ can be found without such elimination.

By differentiating the two given equations, we find $\frac{dy}{dz}$ and $\frac{dz}{dx}$, and from these derivatives, $\frac{dy}{dx}$ may be obtained by the relation

$$\frac{dy}{dx}=\frac{dy}{dz}\,\frac{dz}{dx}. \quad \dots\dots\dots \quad (3)$$

For it is evident that $\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta z}\,\frac{\Delta z}{\Delta x}$,

however small Δx, Δy, and Δz. By taking the limits of the members of this equation we obtain (3). That is, the relation is the same as if the derivatives were ordinary fractions.

For example, suppose

$$\left.\begin{aligned} y&=z^5,\\ z&=a^2-x^2.\end{aligned}\right\} \quad \dots\dots\dots \quad (4)$$

Differentiating these equations, the first with respect to z, and the second with respect to x, we have

$$\frac{dy}{dz}=5z^4, \qquad \frac{dz}{dx}=-2x.$$

By (3), $\quad \frac{dy}{dx}=5z^4(-2x)=-10x(a^2-x^2)^4,\quad$ by (4).

The same result might have been obtained by eliminating z between (4), giving

$$y=(a^2-x^2)^5,$$

and differentiating this with respect to x.

The relation (1) may be obtained as a special case of (3) by substituting $y=x$. This gives

$$\frac{dx}{dz}\frac{dz}{dx}=\frac{dx}{dx}=1.$$

Another form of (3) is

$$\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{dy}{dz}, \qquad \cdots \cdots \qquad (5)$$

which is of frequent use.

EXAMPLES

In Exs. 1–4, find $\frac{dx}{dy}$ and thence $\frac{dy}{dx}$ by (1).

1. $x=\frac{ay-h}{by-k}$, $\qquad \frac{dy}{dx}=\frac{(by-k)^2}{bh-ak}=\frac{bh-ak}{(bx-a)^2}.$

2. $x=\sqrt{1+\sin y}$, $\qquad \frac{dy}{dx}=\frac{2\sqrt{1+\sin y}}{\cos y}=\frac{2}{\sqrt{2-x^2}}.$

3. $x=\frac{y}{1+\log y}$, $\qquad \frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}=\frac{y^2}{xy-x^2}.$

4. $x=a\log\frac{\sqrt{y+a}+\sqrt{y}}{\sqrt{a}}$, $\qquad \frac{dy}{dx}=\frac{2\sqrt{y^2+ay}}{a}=\frac{e^{\frac{2x}{a}}-e^{-\frac{2x}{a}}}{2}.$

In Exs. 5–8 find $\frac{dy}{dz}$ and $\frac{dz}{dx}$, and thence $\frac{dy}{dx}$ by (3).

5. $y=\frac{3z}{2z-1}$, $z=\frac{2x}{3x-2}$, $\qquad \frac{dy}{dx}=\frac{12}{(x+2)^2}.$

6. $y = \log\dfrac{z^2+1}{z}$, $z = e^x$, $\qquad \dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

7. $y = e^z + e^{2z}$, $z = \log(x - x^2)$, $\dfrac{dy}{dx} = 4x^3 - 6x^2 + 1$.

8. $y = \log\dfrac{az+b}{bz+a}$, $z = \sec x + \tan x$,

$$\frac{dy}{dx} = \frac{a^2 - b^2}{2ab + (a^2 + b^2)\cos x}.$$

9. Differentiate $(x^2+2)^2$ with respect to x^3.

Let $y = (x^2+2)^2$, and $z = x^3$. It is required to find $\dfrac{dy}{dz}$.

$$\frac{dy}{dx} = 4x(x^2+2), \qquad \frac{dz}{dx} = 3x^2.$$

By (5) $$\frac{dy}{dz} = \frac{4x(x^2+2)}{3x^2} = \frac{4(x^2+2)}{3x}.$$

10. Find the derivative of $\dfrac{x^3}{a^3} + \dfrac{a^3}{x^3}$ with respect to $\dfrac{x}{a} + \dfrac{a}{x}$.

Ans. $3\left(\dfrac{x^2}{a^2} + 1 + \dfrac{a^2}{x^2}\right)$.

11. Find the derivative of $\sin 3x$ with respect to $\sin x$.

Ans. $3(4\cos^2 x - 3)$.

12. Find the derivative of $\tan^{-1}\sqrt{x}$ with respect to $\log(1+x)$.

Ans. $\dfrac{1}{2\sqrt{x}}$.

13. Find the derivative of $\log\dfrac{a\sin x + b\cos x}{a\sin x - b\cos x}$ with respect to $\dfrac{1}{a^2\sin^2 x - b^2\cos^2 x}$.

Ans. $\dfrac{ab(a^2\tan x - b^2\cot x)}{a^2 + b^2}$.

14. Given $x = 5\cos\phi - \cos 5\phi$, $y = 5\sin\phi - \sin 5\phi$; find $\dfrac{dy}{dx}$.

Ans. $\dfrac{dy}{dx} = \tan 3\phi$

CHAPTER IV

SUCCESSIVE DIFFERENTIATION

57. Definition. As we have seen, the derivative is the result of differentiating a given function of x. This derivative being generally also a function of x, may be again differentiated, and we thus obtain what is called the *second derivative;* the result of three successive differentiations is the *third derivative;* and so on.

For example, if
$$y = x^4,$$
$$\frac{dy}{dx} = 4x^3,$$
$$\frac{d}{dx}\frac{dy}{dx} = 12x^2,$$
$$\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = 24x,$$

58. Notation. The second derivative of y with respect to x, is denoted by $\frac{d^2y}{dx^2}$.

That is,
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}.$$

Similarly,
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{d^2y}{dx^2}.$$
$$\frac{d^4y}{dx^4} = \frac{d}{dx}\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{d^3y}{dx^3}.$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$\frac{d^ny}{dx^n} = \frac{d}{dx}\frac{d^{n-1}y}{dx^{n-1}}.$$

Thus, if

$$y = x^4,$$

$$\frac{dy}{dx} = 4x^3,$$

$$\frac{d^2y}{dx^2} = 12x^2,$$

$$\frac{d^3y}{dx^3} = 24x.$$

The successive derivatives are sometimes called the *first, second, third, ... differential coefficients.*

If the original function of x is denoted by $f(x)$, its successive derivatives are often denoted by

$$f'(x),\quad f''(x),\quad f'''(x),\quad f^{\text{iv}}(x),\quad \cdots\quad f^n(x).$$

59. The *n*th Derivative. It is possible to express the nth derivative of some functions.

For example,

(*a*) From $y = e^{ax}$, we have

$$\frac{dy}{dx} = ae^{ax},\quad \frac{d^2y}{dx^2} = a^2e^{ax},\quad \cdots\quad \frac{d^ny}{dx^n} = a^ne^{ax}.$$

(*b*) From $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$, we have

$$\frac{dy}{dx} = (-1)a(ax+b)^{-2},\quad \frac{d^2y}{dx^2} = (-1)(-2)a^2(ax+b)^{-3},$$

$$\frac{d^3y}{dx^3} = (-1)(-2)(-3)a^3(ax+b)^{-4} = (-1)^3\lfloor\underline{3}\, a^3(ax+b)^{-4},$$

$$\frac{d^ny}{dx^n} = (-1)^n\lfloor\underline{n}\, a^n(ax+b)^{-n-1} = \frac{(-1)^n\lfloor\underline{n}\, a^n}{(ax+b)^{n+1}}.$$

(c) From $y = \sin ax$, we have

$$\frac{dy}{dx} = a \cos ax = a \sin\left(ax + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = a^2 \cos\left(ax + \frac{\pi}{2}\right) = a^2 \sin\left(ax + \frac{2\pi}{2}\right),$$

$$\frac{d^3y}{dx^3} = a^3 \cos\left(ax + \frac{2\pi}{2}\right) = a^3 \sin\left(ax + \frac{3\pi}{2}\right),$$

$$\frac{d^ny}{dx^n} = a^n \sin\left(ax + \frac{n\pi}{2}\right).$$

EXAMPLES

1. $y = 2x^5 - 5x^4 + 20x^3 - 5x^2 + 2x$, $\quad \dfrac{d^3y}{dx^3} = 120(x^2 - x + 1)$.

2. $y = (x^2 - 4)^{\frac{5}{2}}$, $\quad \dfrac{d^2y}{dx^2} = 20(x^2 - 1)(x^2 - 4)^{\frac{1}{2}}$.

3. $y = x^m + x^{-m}$,

$$\frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3} - m(m+1)(m+2)x^{-m-3}$$

4. $y = x^6 + x^9 + x^{12} + x^{15}$, $\quad \dfrac{d^6y}{dx^6} = \lfloor\underline{6} + \dfrac{\lfloor\underline{9}\, x^3}{\lfloor\underline{3}} + \dfrac{\lfloor\underline{12}\, x^6}{\lfloor\underline{6}} + \dfrac{\lfloor\underline{15}\, x^9}{\lfloor\underline{9}}$.

5. $y = x^4 \log x$, $\quad \dfrac{d^5y}{dx^5} = \dfrac{24}{x}$.

6. $y = x^2 \log(x - 1)$, $\quad \dfrac{d^3y}{dx^3} = \dfrac{2(x^2 - 3x + 3)}{(x-1)^3}$.

7. $y = 4(x - 2)e^x + (x - 1)e^{2x}$, $\quad \dfrac{d^2y}{dx^2} = 4x(e^x + e^{2x})$.

8. $x = (t^3 - 3t^2 + \dfrac{9t}{2} - 3)e^{2t}$, $\quad \dfrac{d^2x}{dt^2} = 4t^3e^{2t}$.

9. $r = \log \sec\theta$, $\quad \dfrac{d^4r}{d\theta^4} = 6\sec^4\theta - 4\sec^2\theta$.

10. $y = e^{-x}(11 \sin 2x + 2 \cos 2x)$, $\quad \dfrac{d^3y}{dx^3} = 125 e^{-x} \sin 2x.$

11. $y = \tan^{-1} x$, $\quad \dfrac{d^4y}{dx^4} = \dfrac{24x(1 - x^2)}{(1 + x^2)^4}.$

12. $y = \tan^{-1} \dfrac{e^x - e^{-x}}{2}$, $\quad \dfrac{d^3y}{dx^3} = \dfrac{2(e^{2x} + e^{-2x} - 6)}{(e^x + e^{-x})^3}.$

13. $y = \log \sqrt{x^2 + a^2} + \tan^{-1} \dfrac{x}{a}$, $\quad \dfrac{d^3y}{dx^3} = \dfrac{2(x - a)(x^2 + 4ax + a^2)}{(x^2 + a^2)^3}.$

14. $y = (e^{a\theta} + e^{-a\theta}) \sin a\theta$, $\quad \dfrac{d^4y}{d\theta^4} + 4a^4y = 0.$

15. $y = xe^x(\sin x - \cos x) + 3e^x \cos x$, $\quad \dfrac{d^3y}{dx^3} = 4xe^x \cos x.$

16. $y = e^{-\tan x}$, $\quad \dfrac{d^2y}{dx^2} + (\tan x - 1)^2 \dfrac{dy}{dx} = 0.$

17. $y = \dfrac{\sin nx + \cos nx}{x}$, $\quad \dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + n^2y = 0.$

18. $y = x^2(\sin \log x + \cos \log x)$, $\quad x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = 0.$

19. $y = a^{bx}$, $\quad \dfrac{d^ny}{dx^n} = b^n(\log a)^n a^{bx}.$

20. $y = \log(3x + 2)$, $\quad \dfrac{d^ny}{dx^n} = \dfrac{(-1)^{n-1}3^n \lfloor\!\underline{n-1}}{(3x + 2)^n}.$

21. $y = \sqrt{x + 1}$, $\quad \dfrac{d^ny}{dx^n} = \dfrac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^n(x + 1)^{n-\frac{1}{2}}}.$

22. $y = \sin 5x \sin 2x$, $\quad \dfrac{d^ny}{dx^n} = \dfrac{1}{2}\left[3^n \cos\left(3x + \dfrac{n\pi}{2}\right) - 7^n \cos\left(7x + \dfrac{n\pi}{2}\right)\right].$

The following fractions should be separated into partial fractions before differentiating.

23. $y=\dfrac{1}{x^2-1}$, $\qquad \dfrac{d^ny}{dx^n}=\dfrac{(-1)^n\underline{|n}}{2}\left[\dfrac{1}{(x-1)^{n+1}}-\dfrac{1}{(x+1)^{n+1}}\right].$

24. $y=\dfrac{3x-4}{2x^2+3x-2}$, $\qquad \dfrac{d^ny}{dx^n}=(-1)^n\underline{|n}\left[\dfrac{2}{(x+2)^{n+1}}-\dfrac{2^n}{(2x-1)^{n+1}}\right].$

25. $y=\dfrac{13}{6x^2-5x-6}$, $\qquad \dfrac{d^ny}{dx^n}=(-1)^n\underline{|n}\left[\dfrac{2^{n+1}}{(2x-3)^{n+1}}-\dfrac{3^{n+1}}{(3x+2)^{n+1}}\right].$

26. $y=\dfrac{2x^2+x+1}{2x^2-x-1}=1+\dfrac{2x+2}{2x^2-x-1}$,

$$\frac{d^ny}{dx^n}=(-1)^n\underline{|n}\left[\frac{4}{3(x-1)^{n+1}}-\frac{2^{n+1}}{3(2x+1)^{n+1}}\right].$$

27. $y=\dfrac{x^2}{(x+2)^2}$, $\qquad \dfrac{d^ny}{dx^n}=\dfrac{(-1)^{n+1}4\underline{|n}\,(x-n+1)}{(x+2)^{n+2}}.$

28. $y=\left(\dfrac{ax+1}{ax-1}\right)^2$, $\qquad \dfrac{d^ny}{dx^n}=\dfrac{4(-1)^na^n\underline{|n}\,(ax+n)}{(ax-1)^{n+2}}.$

60. Leibnitz's Theorem. This is a formula for the nth derivative of the product of two factors in terms of the successive derivatives of those factors.

A special case of Leibnitz's Theorem, when $n=1$, is Formula IV.,

$$\frac{d}{dx}(uv)=\frac{du}{dx}v+u\frac{dv}{dx}. \quad \dots\dots\dots\dots \quad (1)$$

For convenience let us use the following abridged notation:

$$v_1=\frac{dv}{dx},\quad v_2=\frac{d^2v}{dx^2},\quad \cdots \quad v_n=\frac{d^nv}{dx^n}.$$

$$u_1=\frac{du}{dx},\quad u_2=\frac{d^2u}{dx^2},\quad \cdots \quad u_n=\frac{d^nu}{dx^n}.$$

Then (1) becomes

$$\frac{d}{dx}(uv) = u_1 v + u v_1. \qquad (2)$$

Differentiating (2),

$$\frac{d^2}{dx^2}(uv) = u_2 v + u_1 v_1 + u_1 v_1 + u v_2 = u_2 v + 2\,u_1 v_1 + u v_2$$

$$\frac{d^3}{dx^3}(uv) = u_3 v + u_2 v_1 + 2\,u_2 v_1 + 2 u_1 v_2 + u_1 v_2 + u v_3$$
$$= u_3 v + 3\,u_2 v_1 + 3 u_1 v_2 + u v_3.$$

We shall find that this law of the terms applies, however far we continue the differentiation, the coefficients being those of the Binomial Theorem; so that

$$\frac{d^n}{dx^n}(uv) = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{\underline{|2}} u_{n-2} v_2 + \cdots + n u_1 v_{n-1} + u v_n. \qquad (3)$$

This may be proved by induction, by showing that, if true for $\frac{d^n}{dx^n}(uv)$, it is also true for $\frac{d^{n+1}}{dx^{n+1}}(uv)$. This exercise is left for the student.

In the ordinary notation (3) becomes

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + n \frac{d^{n-1} u}{dx^{n-1}} \frac{dv}{dx} + \frac{n(n-1)}{\underline{|2}} \frac{d^{n-2} u}{dx^{n-2}} \frac{d^2 v}{dx^2} + \cdots$$
$$+ n \frac{du}{dx} \frac{d^{n-1} v}{dx^{n-1}} + u \frac{d^n v}{dx^n}.$$

EXAMPLES

1. Given $y = x^3 \sin 2x$; find by Leibnitz's Theorem $\frac{d^4 y}{dx^4}$.

From (3), $$\frac{d^4}{dx^4}(uv) = u_4 v + 4\,u_3 v_1 + 6\,u_2 v_2 + 4\,u_1 v_3 + u v_4.$$

$$u = x^3,\ u_1 = 3\,x^2,\ u_2 = 6\,x,\ u_3 = 6,\ u_4 = 0.$$

$$v = \sin 2\,x,\ v_1 = 2 \cos 2\,x,\ v_2 = -4 \sin 2\,x,\ v_3 = -8 \cos 2\,x,$$
$$v_4 = 16 \sin 2\,x.$$

$$\frac{d^4y}{dx^4}=\frac{d^4}{dx^4}(x^3\sin 2x)=0\cdot\sin 2x+4\cdot 6\cdot 2\cos 2x+6\cdot 6x(-4\sin 2x)$$
$$+4\cdot 3x^2(-8\cos 2x)+x^3 16\sin 2x$$
$$=16[(x^3-9x)\sin 2x+(3-6x^2)\cos 2x].$$

2. Given $y=xe^{ax}$; find $\dfrac{d^ny}{dx^n}$.

Here $\quad u=e^{ax},\quad u_1=ae^{ax},\quad\cdots\quad u_{n-1}=a^{n-1}e^{ax},\quad u_n=a^ne^{ax}.$

$\quad v=x,\quad v_1=1,\quad v_2=0,\quad v_3=0,\quad\cdots.$

Substituting in (3), we have

$$\frac{d^ny}{dx^n}=\frac{d^n}{dx^n}(e^{ax}x)=a^ne^{ax}x+na^{n-1}e^{ax}=a^{n-1}e^{ax}(ax+n).$$

3. $y=(x+1)^2\sqrt{x-1},\qquad \dfrac{d^3y}{dx^3}=\dfrac{3(5x^2-14x+13)}{8(x-1)^{\frac{5}{2}}}.$

4. $y=e^x\log x,\qquad \dfrac{d^4y}{dx^4}=e^x\left(\log x+\dfrac{4}{x}-\dfrac{6}{x^2}+\dfrac{8}{x^3}-\dfrac{6}{x^4}\right).$

5. $y=x^3\log(2x+1),\qquad \dfrac{d^4y}{dx^4}=\dfrac{48(x+1)(2x^2+2x+1)}{(2x+1)^4}.$

6. $y=\sin x\log\cos x,\qquad \dfrac{d^4y}{dx^4}=\sin x[\log\cos x-2\tan^2x(3\tan^2x+5)]$

7. $y=x^2a^x,\qquad \dfrac{d^ny}{dx^n}=a^x(\log a)^{n-2}[(x\log a+n)^2-n].$

8. $y=\dfrac{x^2+1}{(x+1)^3},\qquad \dfrac{d^ny}{dx^n}=(-1)^n\,\underline{|n}\,\dfrac{(x-n)^2+n+1}{(x+1)^{n+3}}.$

CHAPTER V

DIFFERENTIALS. INFINITESIMALS

61. The derivative $\frac{dy}{dx}$ has been defined, not as a fraction having a numerator and denominator, but as a single symbol representing the limiting value of $\frac{\Delta y}{\Delta x}$, as Δx approaches zero. In other words, the derivative has not been defined as a ratio, but as the limit of a ratio.

We have seen (Art. 56) that derivatives have certain properties of fractions, and there are some advantages in treating them as such, thus regarding $\frac{dy}{dx}$ as the ratio between dy and dx.

Various definitions have been given for dx and dy, but however defined, they are called *differentials* of x and y respectively. The symbol d before any quantity is read "differential of."

62. Definition of Differential. One definition is the following: The differential of any variable quantity is an *infinitely small increment* in that quantity. That is, dx is an *infinitely small* Δx, and dy an *infinitely small* Δy.

By the direct process (Art. 16) of finding the derivative of an algebraic function, Δy is generally expressed in a series of ascending powers of Δx, beginning with the first.

For example, if $y = x^3$, $\qquad y + \Delta y = (x + \Delta x)^3$,

and $$\Delta y = 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3. \quad . \quad . \quad . \quad (1)$$

In finding the derivative we have

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\,\Delta x + (\Delta x)^2,$$

in which, as Δx approaches zero, the second member approaches $3x^2$ as its limit, the second and third terms approaching the limit zero.

If we let Δx approach zero in equation (1), every term approaches zero, but there is nevertheless a marked distinction between them, in that the second and third terms, containing powers of Δx higher than the first, diminish more rapidly than that term.

Thus we have $\Delta y = 3x^2 \Delta x$ approximately, and the closeness of the approximation increases as Δx approaches zero.

From this point of view, regarding dx and dy as infinitely small increments, we may write

$$dy = 3x^2 dx,$$

not in the sense that both sides ultimately vanish, but in the sense that the *ratio* of the two sides approaches unity.

Thus $dy = 3x^2\,dx$, and $\dfrac{dy}{dx} = 3x^2$,

are two modes of expressing the same relation.

According to the first,

An infinitely small increment of y is $3x^2$ times the corresponding infinitely small increment of x.

According to the second,

The limit of the ratio of the increment of y to that of x, as the latter increment approaches zero, is $3x^2$.

Just as we sometimes say

"An infinitely small arc is equal to its chord," instead of

"The limit of the ratio between an arc and its chord, as these quantities approach zero, is unity."

So in general, if $y = f(x)$,

$$\text{Lim}_{\Delta x = 0} \frac{\Delta y}{\Delta x} = f'(x),$$

that is,

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon,$$

where ϵ approaches zero as Δx approaches zero.

Hence $$\Delta y = f'(x)\,\Delta x + \epsilon\Delta x,$$

and as the term $\epsilon\Delta x$ diminishes more rapidly than the term $f'(x)\Delta x$, we have

$$\Delta y = f'(x)\,\Delta x \text{ approximately,}$$

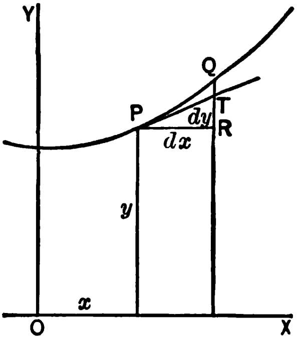
or $$dy = f'(x)\,dx.$$

Corresponding to every equation involving differentials, there is another equation involving derivatives expressing the same relation, and the former may be used as a convenient substitute for the more rigorous statement of the latter.

Thus the use of differentials is not indispensable, but convenient. It should always be kept in mind that their ratio only is important, the derivative being the real subject of mathematical reasoning.

63. Another Definition of Differentials. From $y = f(x)$, we have the derivative $\frac{dy}{dx} = f'(x)$ . . . . (1)

If we regard $\frac{dy}{dx}$ as a fraction, we must define dy and dx so as to satisfy (1); that is, they must be quantities whose ratio is equal to the derivative.



We may assume $dx = \Delta x$,

and consequently $dy = f'(x)\Delta x$.

Graphically, $dx = \Delta x = PR$,

$\Delta y = RQ$,

$f'(x) = \text{slope} = \tan RPT$.

$dy = f'(x)\Delta x = (\tan RPT)PR = RT$.

By this definition of differentials, dx is PR, the increment of x, and dy is then RT, the corresponding increment of the ordinate of the tangent line at P.

That the two definitions are consistent will appear, if we suppose PR to be indefinitely diminished.

The smaller we take PR, the more nearly is $RT = RQ$; that is, the more nearly is $dy = \Delta y$. By regarding PR infinitely small, this definition becomes that of the preceding article.

The second may be said to be the more rigorous of the two definitions, but the first has the advantage of being more symmetrical, and better adapted to the various applications of the calculus to mechanics and physics.

64. Formulæ for Differentials. The formulæ for differentiation may be expressed in the form of differentials by omitting dx in each member.

To each of the formulæ for a derivative, corresponds a formula for a differential.

Thus we have

II. $dc = 0.$

III. $d(u + v) = du + dv.$

IV. $d(uv) = vdu + udv.$

VI. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$

VII. $d(u^n) = nu^{n-1}\,du.$

IX. $d \log u = \frac{du}{u}.$

XI. $de^u = e^u\,du.$

XIII. $d \sin u = \cos u\,du.$

XIV. $d \cos u = -\sin u\,du.$

XV. $d \tan u = \sec^2 u\,du.$

XVI. $d \cot u = -\operatorname{cosec}^2 u\,du.$

XVII. $d \sec u = \sec u \tan u\,du.$

XVIII. $d \operatorname{cosec} u = -\operatorname{cosec} u \cot u\,du.$

XX. $d \sin^{-1} u = \frac{du}{\sqrt{1 - u^2}}.$

XXII. $d \tan^{-1} u = \frac{du}{1 + u^2}.$

XXIV. $d \sec^{-1} u = \frac{du}{u\sqrt{u^2 - 1}}.$

XXVI. $d \operatorname{vers}^{-1} u = \frac{du}{\sqrt{2u - u^2}}.$

Differentiation by the new formulæ is substantially the same as by the old, differing only in using the symbol d instead of $\frac{d}{dx}$.

For example, let $y = \frac{x+3}{x^2+3}$.

$$dy = d\left(\frac{x+3}{x^2+3}\right) = \frac{(x^2+3)\,d(x+3) - (x+3)\,d(x^2+3)}{(x^2+3)^2}$$

$$= \frac{(x^2+3)\,dx - (x+3)\,2\,x dx}{(x^2+3)^2}$$

$$= \frac{(x^2+3-2\,x^2-6\,x)\,dx}{(x^2+3)^2} = \frac{(3-6\,x-x^2)\,dx}{(x^2+3)^2}.$$

If we wish to express the result as the derivative, we have only to divide by dx, giving

$$\frac{dy}{dx} = \frac{3-6\,x-x^2}{(x^2+3)^2}.$$

EXAMPLES

Differentiate the following functions, using differentials in the process:

1. $y = (x-1)(2-3\,x)(2\,x+3)$, $dy = (-18\,x^2+2\,x+11)\,dx.$

2. $x = \frac{(t-1)(t-2)}{(t+1)(t+2)}$, $dx = \frac{6(t^2-2)\,dt}{(t+1)^2(t+2)^2}.$

3. $y = \sqrt{x^2+1}\,\sqrt{x^2-2}$, $dy = \frac{2\,x^3-x}{\sqrt{x^2+1}\,\sqrt{x^2-2}}\,dx.$

4. $r = \frac{\sin^2\theta}{\cos^3\theta}$, $dr = \frac{(2+\sin^2\theta)\sin\theta}{\cos^4\theta}\,d\theta.$

5. $y = e^{\frac{x}{2}}(x^3-6\,x^2+24\,x-40)$, $dy = e^{\frac{x}{2}}\left(\frac{x^3}{2}+4\right)dx.$

6. $r = \sin\theta \log\tan\theta$, $\qquad dr = \cos\theta \log\tan\theta\, d\theta + \sec\theta\, d\theta.$

7. $y = \tan^{-1}\dfrac{4x}{4 - x^2}$, $\qquad dy = \dfrac{4\,dx}{4 + x^2}.$

8. $y = \sin^{-1} 3x + 3x\sqrt{1 - 9x^2}$, $\qquad dy = 6\sqrt{1 - 9x^2}\,dx.$

9. $\phi = \tan^{-1}\tan^3\theta$, $\qquad d\phi = \dfrac{3\tan^2\theta\, d\theta}{\tan^4\theta - \tan^2\theta + 1}.$

65. Order of Infinitesimals. In Art. 62 we have spoken of *infinitely small* or *infinitesimal* increments.

An infinitesimal may be defined as a *variable whose limit is zero.*

If there are several infinitesimals that approach zero simultaneously, one of them, α, may be taken as the standard of comparison and called the *principal infinitesimal.*

Then α^2, α^3, α^n, are said to be infinitesimals of the second, third, nth orders, with respect to α.

In general the order of an infinitesimal is defined as follows: An infinitesimal β is said to be of the nth order with respect to α when

$$\text{Lim}_{\alpha=0}\frac{\beta}{\alpha^n} = k, \text{ a finite quantity, not zero. . . . (1)}$$

When $n = 1$, β is of the first order with respect to α.

When $n = 2$, β is of the second order with respect to α.

From the definition it may be shown that the limit of the ratio of an infinitesimal to one of the same order is finite, and to one of a lower order, zero.

Equation (1), Art. 62, illustrates infinitesimals of different orders. If we write it

$$dy = 3x^2\,dx + 3x(dx)^2 + (dx)^3,$$

and regard dx as the principal infinitesimal, the terms of the second member are infinitesimals of the first, second, and third orders, with respect to dx.

Again, if we regard x as the principal infinitesimal, of what orders are $\sin x$ and $\text{vers}\,x$, with respect to x?

By Art. 12, $\text{Lim}_{x=0}\dfrac{\sin x}{x}=1$, a finite quantity.

Hence by (1) $\sin x$ is an infinitesimal of the first order with respect to x.

$$\text{Lim}_{x=0}\frac{\text{vers}\, x}{x^2}=\text{Lim}_{x=0}\frac{1-\cos x}{\sin^2 x}\,\frac{\sin^2 x}{x^2}=\text{Lim}_{x=0}\frac{1}{1+\cos x}\left(\frac{\sin x}{x}\right)^2=\frac{1}{2}$$

a finite quality.

Hence by (1) $\text{vers}\, x$ is an infinitesimal of the second order with respect to x.

Show that $\tan\theta-\sin\theta$ is an infinitesimal of the third order with respect to θ.

CHAPTER VI

IMPLICIT FUNCTIONS

(See also Art. 114.)

66. In the preceding chapters differentiation has been applied to *explicit* functions of a variable. The same rules or formulæ of differentiation are sufficient for deriving $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \cdots$, when y is an *implicit* function of x; that is, when the relation between y and x is expressed by an equation containing these variables, but not solved with respect to y.

For example, suppose the relation between y and x to be given by the equation

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Differentiating with respect to x,

$$\frac{d}{dx}(a^2y^2 + b^2x^2) = 0,$$

$$2\,a^2y\frac{dy}{dx} + 2\,b^2x = 0,$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

Having thus obtained the first derivative, we may by another differentiation find the second derivative.

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\frac{b^2x}{a^2y} = -\frac{a^2yb^2 - b^2xa^2\frac{dy}{dx}}{a^4y^2} = -\frac{b^2\left(y - x\frac{dy}{dx}\right)}{a^2y^2}.$$

Substituting now for $\frac{dy}{dx}$ its value,

$$\frac{d^2y}{dx^2} = -\frac{b^2(a^2y^2 + b^2x^2)}{a^4y^3} = -\frac{b^4}{a^2y^3}.$$

By differentiating again, we may obtain

$$\frac{d^3y}{dx^3} = -\frac{3\,b^6x}{a^4y^5}.$$

The first differentiation may be conveniently performed by differentials instead of derivatives. Thus we should have from the equation

$$a^2y^2 + b^2x^2 = a^2b^2,$$

$$2\,a^2y\,dy + 2\,b^2x\,dx = 0,$$

giving $$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \text{ as before.}$$

In deriving $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \cdots$, derivatives should be used rather than differentials.

EXAMPLES

Find the following derivatives.

1. $(x-a)^2 + (y-b)^2 = c^2$,

$$\frac{dy}{dx} = -\frac{x-a}{y-b}, \quad \frac{d^2y}{dx^2} = -\frac{c^2}{(y-b)^3}, \quad \frac{d^3y}{dx^3} = -\frac{3c^2(x-a)}{(y-b)^5}.$$

2. $x = y \log(xy)$, $\qquad \frac{dy}{dx} = \frac{xy - y^2}{xy + x^2}.$

3. $(\cos\theta)^\phi = (\sin\phi)^\theta$, $\qquad \frac{d\phi}{d\theta} = \frac{\log\sin\phi + \phi\tan\theta}{\log\cos\theta - \theta\cot\phi}.$

4. $ax^2 + 2\,hxy + by^2 = 1$,

$$\frac{dy}{dx} = -\frac{ax+hy}{hx+by}, \quad \frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}, \quad \frac{d^2x}{dy^2} = \frac{h^2-ab}{(ax+hy)^3}$$

5. $ax^2 + 2hxy + by^2 = 0, \qquad \dfrac{dy}{dx} = \dfrac{y}{x}.$

6. $\tan\theta\tan\phi = m, \dfrac{d\phi}{d\theta} = -\dfrac{\sin 2\phi}{\sin 2\theta}, \dfrac{d^2\phi}{d\theta^2} = \dfrac{2\sin 2\phi(\cos 2\phi + \cos 2\theta)}{\sin^2 2\theta}.$

7. $y - 2x = (x - y)\log(x - y), \quad \dfrac{dy}{dx} = \dfrac{2y - x}{y}, \ \dfrac{d^2y}{dx^2} = -\dfrac{(x-y)^2}{y^3}.$

8. $(3x + y + 6)^3(3y - 3x + 2) = c, \quad \dfrac{dy}{dx} = \dfrac{3x - 2y}{y + 2}.$

9. $r^2\sin^2\theta + 2r + 1 = 0, \qquad \left(\dfrac{dr}{d\theta}\right)^2 + 2r\cot\theta\dfrac{dr}{d\theta} = r^2.$

10. $e^{xy} = a^x b^y, \qquad \dfrac{dy}{dx} = -\dfrac{y - \log a}{x - \log b}, \ \dfrac{d^2y}{dx^2} = \dfrac{2(y - \log a)}{(x - \log b)^2}$

11. $\log(x^2 + y^2) = 2\tan^{-1}\dfrac{y}{x}, \quad \dfrac{dy}{dx} = \dfrac{x + y}{x - y}, \ \dfrac{d^2y}{dx^2} = \dfrac{2x^2 + 2y^2}{(x - y)^3}.$

CHAPTER VII

SERIES. POWER SERIES

67. Convergent and Divergent Series. The series

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

composed of an indefinite number of terms following each other according to some law, is said to be *convergent* when the sum of the terms approaches a finite limit, as the number of terms is indefinitely increased. But when this sum does not approach a finite limit, the series is *divergent.* That is, if S_n denote the sum of the first n terms of (1), the series is *convergent,* when

$$\text{Lim}_{n=\infty} S_n = \text{some definite finite quantity.}$$

When this condition is not satisfied, the series is *divergent.*

Thus the geometrical series,

$$a + ar + ar^2 + ar^3 + \cdots$$

is convergent when r is numerically less than unity, and divergent when r is numerically greater than unity.

For $$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

When $$*|r| < 1, \qquad \text{Lim}_{n=\infty} S_n = \frac{a}{1-r}.$$

When $$|r| > 1, \qquad \text{Lim}_{n=\infty} S_n = \infty.$$

When $|r| = 1$, the series is also divergent.

68. Series of Positive and Negative Terms. Absolute and Conditional Convergence. In the case of series composed of both positive and negative terms, a distinction is made between *absolute convergence* and *conditional convergence.*

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* $|r|$ denotes the numerical value of r.

Before defining these terms, the following theorem should be noticed:

A series whose terms have different signs is convergent if the series formed by taking the absolute values of the terms of the given series is convergent.

Without giving a rigorous proof of the theorem, we may regard the given series as the difference between two series formed of the positive and negative terms respectively.

The theorem is then equivalent to this:

If the sum of two series is convergent, their difference is also convergent.

A series is said to be *absolutely convergent*, when the series of the absolute values of its terms is convergent.

A series whose terms have different signs may be convergent without being absolutely convergent. Such a series is said to be *conditionally convergent.*

For example: $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \qquad (1)$$

converges to the limit $\log_e 2$,
but it is not absolutely convergent, since

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

is divergent (see Art. 70).

Series (1) is accordingly conditionally convergent.

But $$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

is absolutely convergent (see Art. 70).

69. Tests for Convergence. The following are some of the most useful tests.

In every convergent series the nth term must approach zero as a limit, as n is indefinitely increased.

That is, the series $u_1+u_2+u_3+\cdots+u_n+\cdots$

is convergent, only when $\operatorname{Lim}_{n=\infty} u_n=0.$

For $S_n=S_{n-1}+u_n.$

If the sum of the series has a definite limit,

$$\text{Lim}_{n=\infty} S_n = \text{Lim}_{n=\infty} S_{n-1}.$$

Hence $$\text{Lim}_{n=\infty} u_n = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For a *decreasing* series whose terms are alternately positive and negative, this condition is *sufficient.**

For example, $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

is convergent. But the decreasing series

$$\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} \cdots$$

is divergent, as it does not satisfy (1), since $\text{Lim}_{n=\infty} u_n = 1$.

The sum of this series oscillates between two limits, $\log_e 2$ and $1 + \log_e 2$, according as the number of terms is even or odd. Such a series is called an *oscillating* series.

For a series whose terms have the same sign, the condition (1) is *not* sufficient. For example, the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent (see Art. 70).

70. Comparison Test. We may often determine whether a given series of positive terms is convergent or divergent, by comparing its terms with those of another series known to be convergent or divergent.

In this way the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots. \quad . \quad . \quad . \quad . \quad (1)$$

may be shown to be divergent, by comparing it with

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots. \quad . \quad . \quad . \quad . \quad (2)$$

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* The proof of this is omitted.

Each term of (1) is equal to, or greater than, the corresponding term of (2). Hence if (2) is divergent, (1) is also divergent. But (2) may be written

$$1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}+\cdots$$

$$=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

The sum of this series is unlimited; hence (2) is divergent, and therefore (1).

Consider now the more general series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $p=1$, the series (3) becomes (1), which is divergent.

If $p<1$, every term of (3) after the first is greater than the corresponding term of (1). Hence (3) is divergent in this case also.

If $p>1$, compare

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}+\cdots+\frac{1}{15^p}+\cdots \quad . \quad (4)$$

with $$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{8^p}+\cdots+\frac{1}{8^p}+\cdots. \quad . \quad (5)$$

Every term of (4) is equal to, or less than, the corresponding term of (5). But (5) may be written

$$\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\cdots,$$

a geometrical series whose ratio, $\frac{2}{2^p}$, is less than unity.

Hence by Art. 67, (5) is convergent and consequently (4).

Thus it has been shown that

when $p \overset{=}{<} 1$, the series (3) is divergent;
when $p > 1$, the series (3) is convergent.

The series (3) together with the geometrical series are standard series, with which others may often be compared.

71. Cauchy's Ratio Test. This depends upon the ratio of any term to the preceding term. In the series

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots \quad . \quad . \quad . \quad (1)$$

this ratio is $\dfrac{u_{n+1}}{u_n}$.

Let us first consider, from this point of view, the geometrical series

$$a + ar + ar^2 + \cdots + ar^n + ar^{n+1} + \cdots. \quad . \quad . \quad . \quad . \quad (2)$$

Here the ratio $\dfrac{u_{n+1}}{u_n} = r$, and is the same for any two adjacent terms. We have seen (Art. 67) that this series is convergent or divergent, according as

$$|r| < 1, \text{ or } |r| > 1.$$

That is, (2) is convergent or divergent according as

$$\left|\frac{u_{n+1}}{u_n}\right| < 1, \text{ or } \left|\frac{u_{n+1}}{u_n}\right| > 1.$$

If now (1) is any series other than the geometrical series, the ratio $\dfrac{u_{n+1}}{u_n}$ is not constant, but a function of n. The series is then convergent or divergent, according as

$$\text{Lim}_{n=\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1, \text{ or } \text{Lim}_{n=\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1. \quad . \quad . \quad (3)$$

We will first suppose (1) to be a series of positive terms.

Let $$\text{Lim}_{n=\infty}\frac{u_{n+1}}{u_n} = \rho.$$

Suppose $\rho < 1$. By taking n sufficiently large we can make $\dfrac{u_{n+1}}{u_n}$ approach its limit ρ as nearly as we please.

There must be some value m, of n, such that when $n \gtreqless m$,

$$\frac{u_{n+1}}{u_n} < r, \text{ a proper fraction.}$$

Hence $$u_{m+1} < u_m r, \quad u_{m+2} < u_{m+1} r < u_m r^2, \text{ etc.}$$

$$u_m + u_{m+1} + u_{m+2} + \cdots < u_m + u_m r + u_m r^2 + \cdots. \quad . \quad . \quad . \quad (4)$$

But since $r < 1$, the second member of (4), which is the geometrical series, is convergent, and therefore the first member

$$u_m + u_{m+1} + u_{m+2} + \cdots$$

is convergent. Consequently (1) is convergent.

Suppose $\rho > 1$. By similar reasoning, when $n \geqq m$,

$$\frac{u_{n+1}}{u_n} > r, \text{ an improper fraction.}$$

Hence $$u_{m+1} > u_m r, \quad u_{m+2} > u_{m+1} r > u_m r^2 + \text{etc.}$$

$$u_m + u_{m+1} + u_{m+2} + \cdots > u_m + u_m r + u_m r^2 + \cdots.$$

Since $r > 1$, the second member, and therefore the first member, must be divergent.

Thus the theorem is proved for a series of positive terms.

If the terms of (1) have different signs, it is evident from Art. 68 that the series will be absolutely convergent if

$$\text{Lim}_{n=\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

It is also true that for different signs, (1) will be divergent if

$$\text{Lim}_{n=\infty} \left| \frac{u^{n+1}}{u_n} \right| > 1.$$

The proof of this latter statement is omitted.

If $$\text{Lim}_{n=\infty} \left| \frac{u_{n+1}}{u_n} \right| = 1,$$

the series may be either convergent or divergent. There are other tests for such cases, but they will not be considered here.

EXAMPLES

1. Is the following series convergent?

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots + \frac{1}{n\, 2^n} + \cdots.$$

Applying (3), Art. 71, we have $\frac{u_{n+1}}{u_n} = \frac{n}{2(n+1)}$.

$$\text{Lim}_{n=\infty} \frac{n}{2(n+1)} = \frac{1}{2}.$$

As this is less than unity, the given series is convergent.

Its limit is $\log_e 2$, as will appear later.

Determine which of the following series are convergent, and which divergent.

2. $\frac{1}{2}+\frac{2}{\underline{|3}}+\frac{3}{\underline{|4}}+\frac{4}{\underline{|5}}+\cdots.$ By (3), Art. 71

3. $\frac{1}{10}+\frac{\underline{|2}}{10^2}+\frac{\underline{|3}}{10^3}+\frac{\underline{|4}}{10^4}+\cdots.$ By (3), Art. 71

4. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots.$ By (1), Art. 70.

5. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots.$

6. $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\frac{5}{2^4}+\cdots.$

7. $1+\frac{2^2}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\frac{4^4}{\underline{|4}}+\cdots.$

8. $\frac{1}{3}-\frac{2}{5}+\frac{3}{7}-\frac{4}{9}+\frac{5}{11}-\cdots.$ By (1), Art. 69.

9. $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\cdots+\frac{1}{n^2+1}+\cdots.$ Compare with (3), Art. 70.

10. $\frac{1}{1+\sqrt{1}}+\frac{1}{1+\sqrt{2}}+\frac{1}{1+\sqrt{3}}+\cdots.$ Compare with (3), Art. 70.

11. $\log\frac{2}{1}-\log\frac{3}{2}+\log\frac{4}{3}-\log\frac{5}{4}+\cdots.$ By (1), Art. 69

12. $\sec\frac{\pi}{3}-\sec\frac{\pi}{4}+\sec\frac{\pi}{5}-\sec\frac{\pi}{6}+\cdots.$

13. $\sin^2\frac{\pi}{2}+\sin^2\frac{\pi}{3}+\sin^2\frac{\pi}{4}+\sin^2\frac{\pi}{5}+\cdots.$

14. $\dfrac{1+1}{1^2+1}-\dfrac{2+1}{2^2+1}+\dfrac{3+1}{3^2+1}-\dfrac{4+1}{4^2+1}+\cdots.$

15. $\dfrac{1+1}{1^2+1}+\dfrac{2+1}{2^2+1}+\dfrac{3+1}{3^2+1}+\dfrac{4+1}{4^2+1}+\cdots.$

16. $\dfrac{2+1}{2^3-1}+\dfrac{3+1}{3^3-1}+\dfrac{4+1}{4^3-1}+\dfrac{5+1}{5^3-1}+\cdots.$

Answers

Exs. 2, 5, 6, 9, 11, 13, 14, 16, convergent.
Exs. 3, 4, 7, 8, 10, 12, 15, divergent.
Exs. 8, 12, oscillating.

72. Power Series. A series of terms containing the positive integral powers of a variable x, arranged in ascending order, as

$$a_0+a_1x+a_2x^2+a_3x^3+\cdots,$$

is called a *power series in x.* The quantities $a_0, a_1, a_2, \cdots$ are supposed to be independent of x.

For example, $$1+2x+3x^2+4x^3+\cdots,$$

$$1-\frac{y^2}{\underline{|2}}+\frac{y^4}{\underline{|4}}-\frac{y^6}{\underline{|6}}+\cdots,$$

are power series in x and y respectively.

73. Convergence of Power Series. A power series is generally convergent for certain values of the variable and divergent for others.

If we apply the ratio test, (3), Art. 71, to the power series

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots, \quad . \quad . \quad . \quad . \quad (1)$$

we have for the ratio between two terms

$$\frac{u_{n+1}}{u_n}=\frac{a_nx}{a_{n-1}}.$$

$$\text{Lim}_{n=\infty}\left|\frac{u_{n+1}}{u_n}\right|=\text{Lim}_{n=\infty}\left|\frac{a_nx}{a_{n-1}}\right|=|x|\,\text{Lim}_{n=\infty}\left|\frac{a_n}{a_{n-1}}\right|.$$

The series (1) is convergent or divergent according as

$$|x| \operatorname{Lim}_{n=\infty}\left|\frac{a_n}{a_{n-1}}\right| < 1, \text{ or } |x| \operatorname{Lim}_{n=\infty}\left|\frac{a_n}{a_{n-1}}\right| > 1;$$

that is, according as

$$|x| < \operatorname{Lim}_{n=\infty}\left|\frac{a_{n-1}}{a_n}\right|, \text{ or } |x| > \operatorname{Lim}_{n=\infty}\left|\frac{a_{n-1}}{a_n}\right|.$$

The case $|x| = \operatorname{Lim}_{n=\infty}\left|\frac{a_{n-1}}{a_n}\right|$, requires further examination.

For example, consider the series

$$1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + (n+1)x^n + \cdots. \qquad (2)$$

Here $\frac{a_{n-1}}{a_n} = \frac{n}{n+1}$, $\qquad \operatorname{Lim}_{n=\infty}\frac{n}{n+1} = 1.$

Hence (2) is convergent or divergent, according as

$$|x| < 1 \quad \text{or} \quad |x| > 1.$$

We may say that (2) is convergent when $-1 < x < 1$, and the interval from -1 to $+1$ is called the *interval of convergence.*

EXAMPLES

Determine the values of the variable for which the following series are convergent:

1. $1 + x + x^2 + x^3 + \cdots.$

2. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \cdots.$

3. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$

4. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$

5. $x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots.$

6. $1+x+\frac{x^2}{\lfloor\underline{2}}+\frac{x^3}{\lfloor\underline{3}}+\frac{x^4}{\lfloor\underline{4}}+\cdots.$

7. $1-\frac{x^2}{\lfloor\underline{2}}+\frac{x^4}{\lfloor\underline{4}}-\frac{x^6}{\lfloor\underline{6}}+\cdots.$

8. $x-\frac{x^3}{\lfloor\underline{3}}+\frac{x^5}{\lfloor\underline{5}}-\frac{x^7}{\lfloor\underline{7}}+\cdots.$

Answers

Exs. 1–5, convergent when $-1 < x < 1$.

Exs. 6–8, convergent for all values of x.

CHAPTER VIII

EXPANSION OF FUNCTIONS

74. When by any process a given function of a variable is expressed as a power series in that variable, the function is said to be *expanded* into such series.

Thus by ordinary division

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

By the Binomial Theorem

$$(x+a)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4.$$

$$(1-x)^{-2} = 1 + 2\,x + 3\,x^2 + 4\,x^3 + \cdots. \quad . \quad . \quad . \quad . \qquad (2)$$

The methods employed in these expansions are applicable only to functions of a certain kind. We are now about to consider a more general method of expansion, of which the foregoing are only special cases.

It should be noticed that when a function is expanded into a power series of an unlimited number of terms, as (1) and (2), the expansion is valid only for values of x that make the series convergent. For such values, the limit of the sum of the series is the given function, to which we can approximate as closely as we please by taking a sufficient number of terms.

The general method of expansion is known as *Taylor's Theorem* and as *Maclaurin's Theorem.*

These two theorems are so connected that either may be regarded as involving the other. We shall first consider Maclaurin's Theorem.

75. Maclaurin's Theorem. This is a theorem by which a function of x may be expanded into a power series in x. It may be expressed as follows:

$$f(x)=f(0)+f'(0)\frac{x}{1}+f''(0)\frac{x^2}{\underline{|2}}+f'''(0)\frac{x^3}{\underline{|3}}+\cdots,$$

in which $f(x)$ is the given function to be expanded, and $f'(x), f''(x), f'''(x), \cdots$, its successive derivatives.

$f(0), f'(0), f''(0), \cdots$, as the notation implies, denote the values of $f(x), f'(x), f''(x), \cdots$, when $x=0$.

76. Derivation of Maclaurin's Theorem. If we *assume* the possibility of the expansion of $f(x)$ into a power series in x, we may determine the series in the following manner:

Assume

$$f(x)=A+Bx+Cx^2+Dx^3+Ex^4+\cdots, \quad . \quad . \quad . \quad . \quad (1)$$

where $A, B, C, \cdots$ are supposed to be *constant* coefficients.

Differentiating successively, and using the notation just defined, we have

$$f'(x)=B+2\,Cx+3\,Dx^2+4\,Ex^3+\cdots \quad . \quad . \quad . \quad . \quad (2)$$

$$f''(x)=2\,C+2\cdot3\,Dx+3\cdot4\,Ex^2\cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$f'''(x)=2\cdot3\,D+2\cdot3\cdot4\,Ex+\cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$f^{iv}(x)=2\cdot3\cdot4\,E+\cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

Now since equation (1), and consequently (2), (3), $\cdots$ are supposed true for all values of x, they will be true when $x=0$. Substituting zero for x in these equations, we have

from (1), $\quad f(0)=A, \quad A=f(0),$

from (2), $\quad f'(0)=B, \quad B=f'(0),$

from (3), $\quad f''(0)=2\,C, \quad C=\frac{f''(0)}{\underline{|2}},$

from (4), $\quad f'''(0)=2\cdot 3D, \quad D=\dfrac{f'''(0)}{\lfloor\underline{3}},$

from (5), $\quad f^{\text{iv}}(0)=2\cdot 3\cdot 4E, \quad E=\dfrac{f^{\text{iv}}(0)}{\lfloor\underline{4}},$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

Substituting these values of A, B, C, $\cdots$ in (1), we have

$$f(x)=f(0)+f'(0)\frac{x}{1}+f''(0)\frac{x^2}{\lfloor\underline{2}}+f'''(0)\frac{x^3}{\lfloor\underline{3}}+\cdots. \quad . \quad . \quad . \qquad (6)$$

77. As an example in the application of Maclaurin's Theorem, let it be required to expand $\log(1+x)$ into a power series in x.

$$\begin{array}{ll}
f(x)=\log(1+x), & f(0)=\log 1=0. \\
f'(x)=\dfrac{1}{1+x}=(1+x)^{-1}, & f'(0)=1. \\
f''(x)=-(1+x)^{-2}, & f''(0)=-1. \\
f'''(x)=2(1+x)^{-3}, & f'''(0)=2. \\
f^{\text{iv}}(x)=-\lfloor\underline{3}(1+x)^{-4}, & f^{\text{iv}}(0)=-\lfloor\underline{3}. \\
f^{\text{v}}(x)=\lfloor\underline{4}(1+x)^{-5}, & f^{\text{v}}(0)=\lfloor\underline{4}. \\
\cdots \quad \cdots \quad \cdots & \cdots \quad \cdots
\end{array}$$

Substituting in (6), Art. 76, we have

$$\log(1+x)=0+1\cdot x-1\cdot\frac{x^2}{2}+\frac{2x^3}{\lfloor\underline{3}}-\frac{\lfloor\underline{3}\,x^4}{\lfloor\underline{4}}+\frac{\lfloor\underline{4}\,x^5}{\lfloor\underline{5}}-\cdots,$$

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots.$$

78. If in the application of Maclaurin's Theorem to a given function, any of the quantities, $f(0)$, $f'(0)$, $f''(0)$, $\cdots$ are infinite, this function does not admit of expansion in the proposed power series in x.

In this case $f(x)$ or some of its derivatives are discontinuous for $x=0$, and the conditions for Maclaurin's Theorem are not satisfied (see Art. 94).

The functions $\log x$, $\cot x$, $x^{\frac{3}{2}}$, illustrate this case.

EXAMPLES

Expand the following functions into power series by Maclaurin's Theorem:

1. $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots$. Convergent for all values of x.

2. $\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots$. Convergent for all values of x.

3. $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots$. Convergent for all values of x.

4. $(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2$

$$+ \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 + \cdots.$$ Convergent when $|x| < a$.

5. $\log_a(1+x) = \log_a e\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$.

Convergent when $|x| < 1$.

6. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$. Convergent when $|x| < 1$.

7. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. Convergent when $|x| < 1$.

Here

$$f(x) = \tan^{-1}x,$$

$$f'(x) = \frac{1}{1+x^3} = 1 - x^2 + x^4 - x^6 + \cdots,$$

$$f''(x) = -2x + 4x^3 - 6x^5 + \cdots,$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

8. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots$.

Convergent when $|x| < 1$.

Here $$f(x) = \sin^{-1} x,$$
$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}.$$

Expanding by the Binomial Theorem,
$$f'(x) = 1 + ax^2 + bx^4 + cx^6 + \cdots,$$
where $$a = \frac{1}{2},\ b = \frac{1 \cdot 3}{2 \cdot 4},\ c = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6},\ \cdots,$$
$$f''(x) = 2\,ax + 4\,bx^3 + 6\,cx^5 + \cdots,$$
$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

9. $\sin(x+a) = \sin a + x \cos a - \frac{x^2}{|\!\underline{2}} \sin a - \frac{x^3}{|\!\underline{3}} \cos a + \cdots.$
Convergent for all values of x.

10. $\log(1 + x + x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$
Convergent when $|x| < 1$.

11. $e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots.$ Convergent for all values of x.

12. $e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \cdots.$ Convergent for all values of x.

13. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$

14. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots.$

15. $\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots.$

Defining the hyperbolic sine, cosine, and tangent (Art. 277),
$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, show that

16. $\sinh x = x + \frac{x^3}{|\!\underline{3}} + \frac{x^5}{|\!\underline{5}} + \cdots.$

17. $\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \cdots.$

18. $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots.$

19. Show by means of the expansions of Exs. 1, 2, 3, that

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\,\sin x,$$
$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\,\sin x.$$

These are important relations.

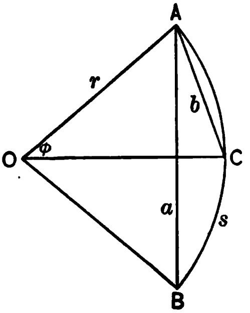
79. Huyghens's Approximate Length of a Circular Arc.

If s denote the length of the arc ACB, a its chord, and b the chord of half the arc, it may be shown that

$s = \frac{8b - a}{3}$, approximately.

Let ϕ be the half angle AOC.

Then $s = 2r\phi$, and by Ex. 2, p. 91,



$$a = 2r\sin\phi = 2r\left(\phi - \frac{\phi^3}{\underline{|3}} + \frac{\phi^5}{\underline{|5}} - \cdots\right),$$

$$b = 2r\sin\frac{\phi}{2} = 2r\left(\frac{\phi}{2} - \frac{\phi^3}{2^3\underline{|3}} + \frac{\phi^5}{2^5\underline{|5}} - \cdots\right).$$

Combining so as to eliminate ϕ^3,

$$8b - a = 2r\left(3\phi - \frac{3\phi^5}{4\underline{|5}} + \cdots\right) = 3s\left(1 - \frac{\phi^4}{4\underline{|5}} + \cdots\right).$$

$$\frac{8b - a}{3} = s\left(1 - \frac{\phi^4}{480} + \cdots\right).$$

If s is an arc of 30°, $\phi = \frac{\pi}{12}$, and the error $< \frac{s}{102000}$.

If s is an arc of 60°, $\phi = \frac{\pi}{6}$, and the error $< \frac{s}{6200}$.

80. Computation by Series.

Compute by Ex. 1, p. 91, $\sqrt{e}$ to 5 decimal places.

Ans. 1.64872.

Compute $\sqrt[10]{e}$ to 10 decimal places. *Ans.* 1.1051709181.

Compute by Ex. 2, p. 91, sin 1° to 8 decimal places. $\pi = 3.14159265$. *Ans.* 0.01745241.

Compute to 4 decimal places the cosine of the angle whose arc is equal to the radius. *Ans.* 0.5403.

81. Calculation of Logarithms. By means of the expansion of $\log(1+x)$, Art. 77, the Napierian logarithms of numbers may be computed.

Let us find the logarithms in the following table.

$$\begin{aligned}\log 2 &= 0.6931,\\ \log 3 &= 1.0986,\\ \log 4 &= 1.3862,\\ \log 5 &= 1.6094,\\ \log 6 &= 1.7917,\\ \log 7 &= 1.9459,\\ \log 8 &= 2.0793,\\ \log 9 &= 2.1972,\\ \log 10 &= 2.3025.\end{aligned}$$

It is only necessary to calculate directly the logarithms of the prime numbers 2, 3, 5, 7, as the others can be expressed in terms of these.

We have from Art. 77,

$$\log 2 = \log(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

This series is convergent, but converges so slowly that 100 terms would give only two decimal places correctly. But we may obtain a series converging much more rapidly by taking

$$\log 2 = \log \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \log(1+\tfrac{1}{3}) - \log(1-\tfrac{1}{3}).$$

For $\log \frac{1+x}{1-x} = \log(1+x) - \log(1-x)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$ Convergent when $|x|<1$.

Thus $$\log 2 = 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \cdots\right).$$

Four terms of this series give $\log 2 = .6931$.

The computation may be arranged as follows:

$$\begin{array}{ll}
\frac{1}{3} = .333333 & \frac{1}{3} = .333333 \\
\frac{1}{3^3} = .037037 & \frac{1}{3\cdot 3^3} = .012346 \\
\frac{1}{3^5} = .004115 & \frac{1}{5\cdot 3^5} = .000823 \\
\frac{1}{3^7} = .000457 & \frac{1}{7\cdot 3^7} = .000065 \\
\frac{1}{3^9} = .000051 & \frac{1}{9\cdot 3^9} = .000006 \\
\cline{2-2}
 & \qquad .34657 \\
 & \qquad\qquad\ \ 2 \\
\cline{2-2}
 & \qquad .69314
\end{array}$$

The numbers in the first column may be obtained by dividing successively by 9.

Any number may be put in the form $\frac{1+x}{1-x}$, and $\log 3 = \log \frac{1+\frac{1}{2}}{1-\frac{1}{2}}$ may be found like log 2.

But having log 2, it is easier to compute

$$\log\frac{3}{2}=\log\frac{1+\frac{1}{5}}{1-\frac{1}{5}},$$

and then $$\log 3=\log\frac{3}{2}+\log 2.$$

Let the student make this computation.

Find log 5 from $$\log\frac{5}{3}=\log\frac{1+\frac{1}{4}}{1-\frac{1}{4}}.$$

In a similar way find log 7 from log 5.

Having obtained the logarithms of 2, 3, 5, 7, find the other logarithms in the table at the beginning of this article.

To obtain the common logarithm, that is, logarithm_{10}, it is only necessary to multiply the Napierian logarithm by .4343, the modulus of the common system.

Find thus the common logarithms of the numbers in the foregoing tables, — first, of 2, 3, 5, 7, and from these the others.

82. Computation of π. From Ex. 7, p. 91, by letting $x=1$, we have

$$\frac{\pi}{4}=\tan^{-1}1=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots,$$

a slowly converging series.

To obtain a series converging more rapidly, we may use

$$\tan^{-1}1=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3},$$

from which $$\frac{\pi}{4}=\frac{1}{2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\frac{1}{7\cdot 2^7}+\cdots$$

$$+\frac{1}{3}-\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}-\frac{1}{7\cdot 3^7}+\cdots.$$

By taking 9 terms of the first series and 5 of the second, the student will find

$$\frac{\pi}{4}=0.463647\cdots+0.321751\cdots$$

and $$\pi=3.14159\cdots.$$

Other forms of $\tan^{-1}1$ may be used, giving series converging even more rapidly, as

$$\tan^{-1}1=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}.$$

$$\tan^{-1}1=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}.$$

By these formulæ the computation has been carried to 200 decimal places.

83. Taylor's Theorem. This is a theorem for expanding a function of the sum of two quantities into a power series in one of these quantities.

As the Binomial Theorem expands $(x+h)^n$ into a power series in h, so Taylor's Theorem expands $f(x+h)$ into such a series. It may be expressed as follows:

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{\underline{|2}}+f'''(x)\frac{h^3}{\underline{|3}}+\cdots.$$

84. The proof of Taylor's Theorem depends upon the following principle:

If we differentiate $f(x+h)$ with respect to x, regarding h constant, the result is the same as if we differentiate it with respect to h, regarding x constant.

That is, $$\frac{d}{dx}f(x+h)=\frac{d}{dh}f(x+h).$$

For, let $$z=x+h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then by (3), Art. 56,

$$\frac{d}{dx}f(x+h)=\frac{d}{dx}f(z)=\frac{d}{dz}f(z)\frac{dz}{dx},$$

$$\frac{d}{dh}f(x+h)=\frac{d}{dh}f(z)=\frac{d}{dz}f(z)\frac{dz}{dh}.$$

But from (1), $\frac{dz}{dx}=1$, and $\frac{dz}{dh}=1$;

therefore $$\frac{d}{dx}f(x+h)=\frac{d}{dh}f(x+h).$$

85. Derivation of Taylor's Theorem. If we *assume* the possibility of the expansion of $f(x+h)$ into a power series in h, we may determine the series by the aid of the preceding article. Assume

$$f(x+h)=A+Bh+Ch^2+Dh^3+\cdots \quad . \quad . \quad . \quad . \quad (1)$$

where A, B, C, $\cdots$ are supposed to be functions of x but not of h.

Differentiating (1), first with respect to x, then with respect to h,

$$\frac{d}{dx}f(x+h)=\frac{dA}{dx}+\frac{dB}{dx}h+\frac{dC}{dx}h^2+\frac{dD}{dx}h^3+\cdots,$$

$$\frac{d}{dh}f(x+h)=B+2\,Ch+3Dh^2+\cdots.$$

By Art. 84, the first members of these two equations are equal to each other, therefore

$$\frac{dA}{dx}+\frac{dB}{dx}h+\frac{dC}{dx}h^2+\cdots=B+2\,Ch+3\,Dh^2+\cdots.$$

Equating the coefficients of like powers of h according to the principle of Undetermined Coefficients, we have

$$\frac{dA}{dx}=B, \qquad\qquad B=\frac{dA}{dx}.$$

$$\frac{dB}{dx}=2C, \qquad\qquad C=\frac{1}{2}\frac{d^2A}{dx^2}.$$

$$\frac{dC}{dx}=3D, \qquad\qquad D=\frac{1}{\underline{|3}}\frac{d^3A}{dx^3}.$$

$$\cdots \quad \cdots \qquad\qquad \cdots \quad \cdots$$

The coefficient A may be found from (1) by putting $h=0$, as that equation is supposed true for all values of h.

Then $$A=f(x).$$

Hence $$B=\frac{dA}{dx}=f'(x).$$

$$C=\frac{1}{2}\frac{d^2A}{dx^2}=\frac{1}{2}f''(x).$$

$$D=\frac{1}{\underline{|3}}\frac{d^3A}{dx^3}=\frac{1}{\underline{|3}}f'''(x).$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots$$

Substituting these expressions for A, B, C, $\cdots$ in (1), we have

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{\underline{|2}}+f'''(x)\frac{h^3}{\underline{|3}}+\cdots. \quad . \quad , \qquad (2)$$

86. Maclaurin's Theorem may be obtained from Taylor's Theorem by substituting $x=0$. We then have

$$f(h)=f(0)+f'(0)h+f''(0)\frac{h^2}{\underline{|2}}+f'''(0)\frac{h^3}{\underline{|3}}+\cdots.$$

This is Maclaurin's Theorem expressed in terms of h instead of x.

87. As an example in the application of Taylor's Theorem, let it be required to expand $\sin(x+h)$ into a power series in h.

$$f(x+h)=\sin(x+h);$$

hence
$$\begin{aligned}f(x)&=\sin x,\\ f'(x)&=\cos x,\\ f''(x)&=-\sin x,\\ f'''(x)&=-\cos x,\\ f^{\text{iv}}(x)&=\sin x.\\ \cdots\quad&\cdots\quad\cdots\end{aligned}$$

Substituting these expressions in (2), Art. 85, we find

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{\underline{|2}}\sin x-\frac{h^3}{\underline{|3}}\cos x+\frac{h^4}{\underline{|4}}\sin x+\cdots.$$

EXAMPLES

Derive the following expansions by Taylor's Theorem:

1. $\cos(x+h)=\cos x - h\sin x - \frac{h^2}{\underline{|2}}\cos x + \frac{h^3}{\underline{|3}}\sin x + \cdots$

2. $e^{x+h}=e^x\left(1+h+\frac{h^2}{\underline{|2}}+\frac{h^3}{\underline{|3}}+\cdots\right).$

3. $(x+h)^7=x^7+7x^6h+\cdots.$

4. $(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{\underline{|2}}x^{n-2}h^2+\cdots.$

5. $\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\frac{h^4}{4x^4}+\cdots.$

6. $\tan(x+h)=\tan x+h\sec^2 x+h^2\sec^2 x\tan x$
$+\frac{h^3}{3}(3\sec^4 x-2\sec^2 x)+\cdots.$

7. Compute from Ex. 1, $\cos 62° = 0.4695$.

8. Compute from Ex. 6, $\tan 44° = 0.9657$, $\tan 46° = 1.0355$.

9. $\frac{f(x+h)+f(x-h)}{2}=f(x)+\frac{h^2}{\underline{|2}}f''(x)+\frac{h^4}{\underline{|4}}f^{\text{iv}}(x)+\cdots.$

10. $\frac{f(x+h)-f(x-h)}{2}=hf'(x)+\frac{h^3}{\underline{|3}}f'''(x)+\frac{h^5}{\underline{|5}}f^{\text{v}}(x)+\cdots.$

As a special case of Ex. 10, derive

$$\frac{1}{2}\log\frac{x+h}{x-h}=\frac{h}{x}+\frac{h^3}{3x^3}+\frac{h^5}{5x^5}+\cdots.$$

11. $f(2x)=f(x)+xf'(x)+\frac{x^2}{\underline{|2}}f''(x)+\frac{x^3}{\underline{|3}}f'''(x)+\cdots.$

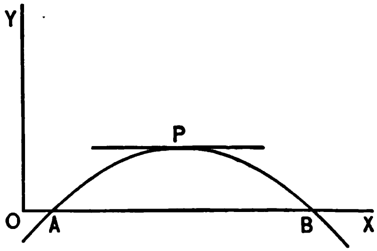
12. $f\left(\frac{x^2}{1+x}\right)=f(x)-\frac{x}{1+x}f'(x)+\frac{x^2}{(1+x)^2}\frac{f''(x)}{\underline{|2}}$
$-\frac{x^3}{(1+x)^3}\frac{f'''(x)}{\underline{|3}}+\cdots.$

13. If $y=f(x)$, show that

$$\Delta y=\frac{dy}{dx}\Delta x+\frac{d^2y}{dx^2}\frac{(\Delta x)^2}{\underline{|2}}+\frac{d^3y}{dx^3}\frac{(\Delta x)^3}{\underline{|3}}+\cdots.$$

88. In the preceding derivations of Taylor's and Maclaurin's Theorems, the possibility of the expansion in the proposed form has been assumed. In the remainder of this chapter we shall show how Taylor's Theorem may be derived without such assumption.

89. Rolle's Theorem. If a given function $\phi(x)$ is zero when $x=a$ and when $x=b$, and is continuous between those values, as well as its derivative $\phi'(x)$; then $\phi'(x)$ must be zero for some value of x between a and b.



Let the function be represented by the curve $y=\phi(x)$. Let $OA=a$, $OB=b$. Then according to the hypothesis, $y=0$ when $x=a$, and when $x=b$.

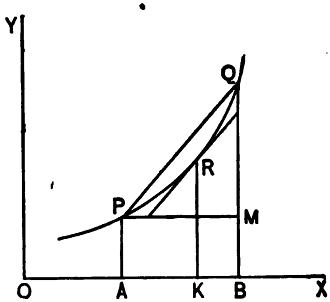
Since the curve is continuous between A and B, there must be some point P between them, where the tangent is parallel to OX, and consequently $\phi'(x)=0$.

90. Mean Value Theorem. If $f(x)$ is continuous from $x=a$ to $x=b$, there must be some value x_1 of x, between a and b, for which

$$\frac{f(b)-f(a)}{b-a}=f'(x_1).$$

This may be stated geometrically thus:

The difference of the ordinates of two points of a continuous curve, divided by the corresponding difference of abscissas of these points, equals the slope of the curve at some intermediate point.



In the figure let the curve PRQ represent $y=f(x)$.

Let $OA = a$, $OB = b$. Then

$$\frac{f(b)-f(a)}{b-a} = \frac{QM}{PM} = \tan QPM.$$

At some point of the curve, as R, between P and Q, a tangent can be drawn parallel to PQ. Call $OK = x_1$. Then the slope of the tangent at R is $f'(x_1)$, which equals $\tan QPM$.

Hence $\dfrac{f(b)-f(a)}{b-a} = f'(x_1)$, where $a < x_1 < b$. . (1)

If we let $AB = b - a = h$, $b = a + h$, (1) may be written

$$f(a+h) = f(a) + hf'(a+\phi h), \quad \text{where} \quad 0 < \phi < 1. \quad . \ . \quad (2)$$

91. Another Proof. The following method of deriving (2), Art. 90, is important, in that it may be extended to higher derivatives of $f(x)$, as appears in Arts. 92, 93.

Let R be defined by

$$f(a+h) - f(a) - hR = 0. \quad . \ . \ . \ . \ . \quad (1)$$

That is, let R denote $\dfrac{f(a+h)-f(a)}{h}$.

Consider a function of x whose expression is the same as (1) with x substituted for h. Call this function $\phi(x)$.

That is, $\phi(x) = f(a+x) - f(a) - xR$. . . . . (2)

Differentiating, $\phi'(x) = f'(a+x) - R$. . . . . . . (3)

It is evident from (2) that $\phi(x) = 0$, when $x = h$, by (1);

also $\phi(x) = 0$, when $x = 0$.

Hence by Rolle's Theorem, Art. 89, $\phi'(x) = 0$, for some value of x between 0 and h.

Calling this value of x, θh, we have from (3)

$$f'(a+\theta h) - R = 0.$$

Substituting this value of R in (1),

$$f(a+h) = f(a) + hf'(a+\theta h).$$

92. Extension of Mean Value Theorem. We may extend the method of the preceding article so as to include the second derivative, and obtain
$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(a+\theta h).$$

Define R by
$$f(a+h)-f(a)-hf'(a)-\frac{h^2}{2}R=0. \quad . \quad . \quad . \quad . \quad (1)$$

Let
$$\phi(x)=f(a+x)-f(a)-xf'(a)-\frac{x^2}{2}R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence
$$\phi'(x)=f'(a+x)-f'(a)-xR,$$
$$\phi''(x)=f''(a+x)-R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) it is evident that $\phi(x)=0$, when $x=h$, by (1);

also $\phi(x)=0$, when $x=0$.

Hence by Rolle's Theorem, Art. 89, $\phi'(x)=0$, for some value, x_1, of x, between 0 and h.

Also $\phi'(x)=0$, when $x=0$.

Hence $\phi''(x)=0$, for some value, x_2, between 0 and x_1, that is, between 0 and h. Writing $x_2=\theta h$, we have from (3),
$$f''(a+\theta h)-R=0.$$

Substituting this value of R in (1), we have
$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(a+\theta h).$$

It is to be noticed that it is assumed that $f(x)$, $f'(x)$, and $f''(x)$ are continuous from $x=a$ to $x=a+h$.

93. Taylor's Theorem. This may now be derived by extending the preceding method so as to include the nth derivative.

It is assumed that $f(x)$ and its first n derivatives are continuous from $x=a$ to $x=a+h$.

Define R by
$$f(a+h)-f(a)-hf'(a)-\frac{h^2}{\underline{|2}}f''(a)-\cdots-\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(a)-\frac{h^n}{\underline{|n}}R=0. \quad (1)$$

Let

$$\phi(x)=f(a+x)-f(a)-xf'(a)-\frac{x^2}{\lfloor\underline{2}}f''(a)-\cdots-\frac{x^{n-1}}{\lfloor\underline{n-1}}f^{n-1}(a)-\frac{x^n}{\lfloor\underline{n}}R,$$

$$\phi'(x)=f'(a+x)-f'(a)-xf''(a)-\cdots-\frac{x^{n-2}}{\lfloor\underline{n-2}}f^{n-1}(a)-\frac{x^{n-1}}{\lfloor\underline{n-1}}R,$$

$$\phi''(x)=f''(a+x)-f''(a)-\cdots-\frac{x^{n-3}}{\lfloor\underline{n-3}}f^{n-1}(a)-\frac{x^{n-2}}{\lfloor\underline{n-2}}R,$$

. . . .

$$\phi^{n-1}(x)=f^{n-1}(a+x)-f^{n\ 1}(a)-xR,$$

$$\phi^n(x)=f^n(a+x)-R. \qquad (2)$$

As in the preceding articles, it is evident that

$\phi(x)=0$, when $x=h$, and also when $x=0$.

Hence $\phi'(x)=0$, when $x=x_1$, where $0<x_1<h$.

But $\phi'(x)=0$, when $x=0$; hence

$\phi''(x)=0$, when $x=x_2$, where $0<x_2<x_1$.

Continuing this reasoning, we find

$\phi^n(x)=0$, when $x=x_n$, where $0<x_n<x_{n-1}$,

that is, where x_n is between 0 and h.

Hence from (2) $\phi^n(\theta h)=f^n(a+\theta h)-R=0$.

Substituting this value of R in (1), we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(a)+\cdots+\frac{h^{n-1}}{\lfloor\underline{n-1}}f^{n-1}(a)+\frac{h^n}{\lfloor\underline{n}}f^n(a+\theta h).$$

Since a is *any* quantity, we may write x in place of a, giving

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor\underline{2}}f''(x)+\cdots+\frac{h^{n-1}}{\lfloor\underline{n-1}}f^{n-1}(x)$$

$$+\frac{h^n}{\lfloor\underline{n}}f^n(x+\theta h). \qquad (3)$$

94. Remainder. The last term of this equation

$$\frac{h^n}{\lfloor n}f^n(x+\theta h)$$

is called the *remainder after n terms in Taylor's Theorem.* When the limit of this remainder is zero, as n is indefinitely increased, Taylor's Theorem gives a convergent series.

We have already seen (Art. 86) that Maclaurin's Theorem is a special case of Taylor's Theorem, so that corresponding to (3) of the preceding article, we may write Maclaurin's Theorem

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)+\cdots+\frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0)+\frac{x^n}{\lfloor n}f^n(\theta x),$$

$f(x)$ and its first n derivatives being assumed continuous for values from 0 to x.

Thus the *remainder after n terms in Maclaurin's Theorem* is

$$\frac{x^n}{\lfloor n}f^n(\theta x).$$

When $$\text{Lim}_{n=\infty}\left[\frac{x^n}{\lfloor n}f^n(\theta x)\right]=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Maclaurin's Theorem gives a convergent series.

Applying (1) to $f(x)=e^x$, we have

$$\text{Lim}_{n=\infty}\left[\frac{x^n}{\lfloor n}e^{\theta x}\right]=0,$$

which is evidently satisfied for all values of x.

The same is true for $f(x)=\sin x$, and $f(x)=\cos x$.

If $f(x)=\log(1+x)$, (1) becomes

$$\text{Lim}_{n=\infty}\left[\frac{x^n}{\lfloor n}\,\frac{(-1)^{n-1}\lfloor n-1}{(1+\theta x)^n}\right]$$

$$=\text{Lim}_{n=\infty}\left[\frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n\right]=0.$$

This is satisfied when $|x|<1$.

It is to be noticed that the preceding test for convergence is of no practical use, unless the nth derivative of $f(x)$ can be expressed.

CHAPTER IX

INDETERMINATE FORMS

95. Value of Fraction as Limit. The value of the fraction $\frac{\phi(x)}{\psi(x)}$ for any assigned value of x, as $x = a$, is $\frac{\phi(a)}{\psi(a)}$.

This is a definite quantity, unless $\phi(a)$ or $\psi(a)$ is zero or infinity. When this is the case, we may, by regarding the fraction as a continuous variable, define its value when x *equals* a, as its limit when x *approaches* a.

That is, the value of $\frac{\phi(x)}{\psi(x)}$, when $x = a$, is defined to be

$$\text{Lim}_{x=a} \frac{\phi(x)}{\psi(x)}, \text{ or what is the same thing,}$$

$$\text{Lim}_{h=0} \frac{\phi\,(a+h)}{\psi\,(a+h)}.$$

There is no difficulty in determining this limit immediately, when the numerator only, or the denominator only, is zero or infinity; or when one is zero and the other infinity.

We will now consider the cases where, for some assigned value of x, the numerator and denominator are both zero or both infinity. The fraction is then said to be *indeterminate*.

96. Evaluation of the Indeterminate Form $\frac{0}{0}$. Frequently a transformation of the given fraction will determine its value.

Thus, $$\frac{x^2+x-2}{x^2-1} = \frac{0}{0}, \text{ when } x = 1.$$

But if we reduce the fraction to its lowest terms, we have

$$\text{Lim}_{x=1} \frac{x^2+x-2}{x^2-1} = \text{Lim}_{x=1} \frac{x+2}{x+1} = \frac{3}{2}.$$

Again, $$\frac{x-2}{\sqrt{x-1}-1}=\frac{0}{0}, \text{ when } x=2.$$

By rationalizing the denominator,

$$\text{Lim}_{x=2}\frac{x-2}{\sqrt{x-1}-1}=\text{Lim}_{x=2}\frac{(x-2)(\sqrt{x-1}+1)}{x-2}$$

$$=\text{Lim}_{x=2}(\sqrt{x-1}+1)=2.$$

As another illustration,

$$\frac{\cos 2\theta}{\cos\theta-\sin\theta}=\frac{0}{0}, \text{ when } \theta=\frac{\pi}{4}.$$

But $$\text{Lim}_{\theta=\frac{\pi}{4}}\frac{\cos 2\theta}{\cos\theta-\sin\theta}=\text{Lim}_{\theta=\frac{\pi}{4}}\frac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}$$

$$=\text{Lim}_{\theta=\frac{\pi}{4}}(\cos\theta+\sin\theta)=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}=\sqrt{2}.$$

The Differential Calculus furnishes the following general method:

97. *Form a new fraction, taking the derivative of the given numerator for a new numerator, and of the given denominator for a new denominator. The value of this new fraction, for the assigned value of the variable, is the limiting value of the given fraction.*

We will now show how this rule is derived.

Suppose the fraction $\frac{\phi(x)}{\psi(x)}=\frac{0}{0}$, when $x=a$; that is, $\phi(a)=0$, and $\psi(a)=0$.

By Art. 95 the required value of the fraction is the limit of $\frac{\phi(a+h)}{\psi(a+h)}$, as h approaches zero.

By the Mean Value Theorem, (2), Art. 90

$$\phi(a+h)=\phi(a)+h\phi'(a+\theta h),$$

$$\psi(a+h)=\psi(a)+h\psi'(a+\theta_1 h),$$

where θ and θ_1 are proper fractions.

But since $\phi(a)=0$ and $\psi(a)=0$, we have

$$\frac{\phi(a+h)}{\psi(a+h)}=\frac{h\phi'(a+\theta h)}{h\psi'(a+\theta_1 h)}=\frac{\phi'(a+\theta h)}{\psi'(a+\theta_1 h)}.$$

Hence
$$\text{Lim}_{h=0}\frac{\phi(a+h)}{\psi(a+h)}=\frac{\phi'(a)}{\psi'(a)},$$

which is the theorem expressed by the rule.

If $\phi'(a)=0$ and $\psi'(a)=0$, it follows likewise that

$$\text{Lim}_{h=0}\frac{\phi'(a+h)}{\psi'(a+h)}=\frac{\phi''(a)}{\psi''(a)};$$

that is, the process expressed by the rule must be repeated, and as often as may be necessary to obtain a result which is not indeterminate.

For example, let us find the limiting value of the fraction in Art. 96.

$$\frac{\phi(x)}{\psi(x)}=\frac{x^2+x-2}{x^2-1}=\frac{0}{0}, \text{ when } x=1.$$

$$\frac{\phi'(x)}{\psi'(x)}=\frac{2x+1}{2x}=\frac{3}{2}, \text{ when } x=1.$$

Thus the required limiting value is $\frac{3}{2}$.

For another example, let us find the limiting value, when $x=0$.

of
$$\frac{e^x+e^{-x}-2}{1-\cos x}.$$

$$\frac{\phi(x)}{\psi(x)}=\frac{e^x+e^{-x}-2}{1-\cos x}=\frac{0}{0}, \text{ when } x=0.$$

$$\frac{\phi'(x)}{\psi'(x)}=\frac{e^x-e^{-x}}{\sin x}=\frac{0}{0}, \qquad \text{when } x=0.$$

$$\frac{\phi''(x)}{\psi''(x)}=\frac{e^x+e^{-x}}{\cos x}=2, \qquad \text{when } x=0.$$

Thus the required limiting value is 2

EXAMPLES

Find the limiting values of the following fractions for the assigned values of the variable.

1. $\dfrac{x(x-1)^n-2}{x^2-2x}$, when $x=2$. *Ans.* $n+\dfrac{1}{2}$.

2. $\dfrac{\log(3x^2+x-3)}{\log x}$, when $x=1$. *Ans.* 7.

3. $\dfrac{a^x-1}{b^x-1}$, when $x=0$. *Ans.* $\log_b a$.

4. $\dfrac{x-\tan^{-1}x}{x-\sin^{-1}x}$, when $x=0$. *Ans.* -2.

5. $\dfrac{\log(x^2-4x+5)}{\log\cos(x-2)}$, when $x=2$. *Ans.* -2.

6. $\dfrac{xe^x-\log(x+1)}{x^2}$, when $x=0$. *Ans.* $\dfrac{3}{2}$.

7. $\dfrac{\theta \operatorname{vers}\theta}{\sin^2\theta}$, when $\theta=0$. *Ans.* 0.

8. $\dfrac{\sin m\alpha-\sin n\alpha}{\sin(m-n)\alpha}$, when $m=n$. *Ans.* $\cos n\alpha$.

9. $\dfrac{\sin\left(\theta+\dfrac{\pi}{4}\right)-1}{\log\sin 2\theta}$, when $\theta=\dfrac{\pi}{4}$. *Ans.* $\dfrac{1}{4}$.

10. $\dfrac{a^a-b^b}{a^b-b^a}$, when $a=b$. *Ans.* $\dfrac{1+\log b}{1-\log b}$.

11. $\dfrac{\log_b a-\log_a b}{a-b}$, when $a=b$. *Ans.* $\dfrac{2}{b\log b}$.

12. $\dfrac{x^5-2x^3-4x^2+9x-4}{x^4-2x^3+2x-1}$, when $x=1$. *Ans.* 4.

13. $\dfrac{\tan nx - n\tan x}{n\sin x - \sin nx}$, when $x = 0$. *Ans.* 2.

14. $\dfrac{\tan nx - n\tan x}{n\sin x - \sin nx}$, when $n = 1$. *Ans.* $\dfrac{x\sec^2 x - \tan x}{\sin x - x\cos x}$

15. $\dfrac{m\sin x - \sin mx}{(x-1)e^x + (x+1)e^{-x}}$, when $x = 0$. *Ans.* $\dfrac{m^3 - m}{4}$.

16. $\dfrac{e^{2x} + e^{-2x} - 4x^2 - 2}{\log\sec^2 x - x^2}$, when $x = 0$. *Ans.* 8.

98. Evaluation of the Indeterminate Form $\frac{\infty}{\infty}$. The method is the same as that given in Art. 97 for the form $\frac{0}{0}$.

It has been shown in that article that

$$\operatorname{Lim}_{h=0}\frac{\phi(a+h)}{\psi(a+h)} = \frac{\phi'(a)}{\psi'(a)}, \quad \ldots\ldots \quad (1)$$

if $\phi(a) = 0$, and $\psi(a) = 0$.

It may be shown that (1) is true also, if $\phi(a) = \infty$ and $\psi(a) = \infty$. For the proof of this the student is referred to more extensive treatises on the Differential Calculus.

For example, find the limiting value of $\dfrac{\log x}{\cot x}$, when $x = 0$.

$$\frac{\phi(x)}{\psi(x)} = \frac{\log x}{\cot x} = \frac{\infty}{\infty}, \text{ when } x = 0.$$

$$\frac{\phi'(x)}{\psi'(x)} = \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = -\frac{\sin^2 x}{x} = \frac{0}{0}, \text{ when } x = 0.$$

$$\frac{\phi''(x)}{\psi''(x)} = -\frac{2\sin x\cos x}{1} = \frac{0}{1} = 0, \quad \text{when } x = 0.$$

Thus the required limiting value is 0.

NOTE. — The form $\frac{\infty}{\infty}$ can in most cases be avoided by transformation into $\frac{0}{0}$.

99. Evaluation of the Indeterminate Forms $0 \cdot \infty$, $\infty - \infty$. Transform the expression into a fraction, which will assume either the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

For example, find the value of

$$(\pi - 2x) \tan x, \text{ when } x = \frac{\pi}{2}.$$

This takes the form $0 \cdot \infty$.

But $$(\pi - 2x) \tan x = \frac{\pi - 2x}{\cot x} = \frac{0}{0}, \text{ when } x = \frac{\pi}{2}.$$

$$\frac{\phi'(x)}{\psi'(x)} = \frac{-2}{-\operatorname{cosec}^2 x} = 2, \text{ when } x = \frac{\pi}{2}.$$

Thus the required limiting value is 2.

For another example, find the limiting value of

$$\frac{1}{\log x} - \frac{1}{x-1}, \text{ when } x = 1.$$

This takes the form $\infty - \infty$.

But $$\frac{1}{\log x} - \frac{1}{x-1} = \frac{x - 1 - \log x}{(x-1)\log x} = \frac{0}{0}, \text{ when } x = 1.$$

$$\frac{\phi'(x)}{\psi'(x)} = \frac{1 - \frac{1}{x}}{1 - \frac{1}{x} + \log x} = \frac{0}{0}, \text{ when } x = 1.$$

$$\frac{\phi''(x)}{\psi''(x)} = \frac{\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{2}, \text{ when } x = 1.$$

Thus the required limiting value is $\frac{1}{2}$.

100. Evaluation of the Exponential Indeterminate Forms, $0^0, 1^\infty, \infty^0$. Take the logarithm of the given expression, which will have the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The limiting value of this logarithm will determine that of the given function.

For example, find the limiting value of x^x, when $x=0$. This takes the form 0^0.

Let $$y = x^x;$$

then $$\log y = x \log x = \frac{\log x}{x^{-1}} = \frac{\infty}{\infty}, \text{ when } x = 0.$$

$$\frac{\phi'(x)}{\psi'(x)} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0, \text{ when } x = 0.$$

Thus the limiting value of $\log y$ is 0.

Hence the limiting value of y is $e^0 = 1$.

For another example find the limiting value of

$$(1+ax)^{\frac{1}{x}}, \text{ when } x=0.$$

This takes the form of 1^∞.

Let $$y = (1+ax)^{\frac{1}{x}},$$

$$\log y = \frac{\log(1+ax)}{x} = \frac{0}{0}, \text{ when } x = 0.$$

$$\frac{\phi'(x)}{\psi'(a)} = \frac{\frac{a}{1+ax}}{1} = a, \text{ when } x = 0.$$

The limiting value of $\log y$ being a, the limiting value of y is e^a.

EXAMPLES

Find the limiting values of the following expressions for the assigned values of the variable.

1. $\frac{1}{x}-\frac{1}{x^2}\log(1+x)$, when $x=0$. *Ans.* $\frac{1}{2}$.

2. $2^x\tan\frac{3}{2^x}$, when $x=\infty$. *Ans.* 3.

3. $x\tan x-\frac{\pi}{2}\sec x$, when $x=\frac{\pi}{2}$. *Ans.* -1.

4. $\frac{\log\tan ax}{\log\tan bx}$, when $x=0$. *Ans.* 1.

5. $\frac{1}{2x^2}-\frac{1}{2x\tan x}$, when $x=0$. *Ans.* $\frac{1}{6}$.

6. $\frac{\sec 3x}{\sec 5x}$, when $x=\frac{\pi}{2}$. *Ans.* $-\frac{5}{3}$.

7. $\left(\frac{x^2+x}{2}\right)^{\frac{1}{x-1}}$, when $x=1$. *Ans.* $e^{\frac{3}{2}}$.

8. $(\operatorname{cosec}\theta)^{\tan^2\theta}$ when $\theta=\frac{\pi}{2}$. *Ans.* $\sqrt{e}$.

9. $(\tan\theta)^{\cos\theta}$ when $\theta=\frac{\pi}{2}$. *Ans.* 1.

10. $\left(\frac{2\sec^2\theta-1}{3}\right)^{\tan 2\theta}$ when $\theta=\frac{\pi}{4}$. *Ans.* $e^{-\frac{4}{\pi}}$

11. $\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{3}{x}}$, when $x=0$. *Ans.* abc.

12. $(\log x)^{\frac{1}{x-e}}$, when $x=e$. *Ans.* $e^{\frac{1}{e}}$.

13. $(a^x+x)^{\frac{1}{x}}$, when $x=0$. *Ans.* ae.

14. $\left(\frac{\cos ax+\cos bx}{2}\right)^{\frac{4}{x^2}}$, when $x=0$. *Ans.* $\frac{1}{e^{a^2+b^2}}$

CHAPTER X

MAXIMA AND MINIMA OF FUNCTIONS OF ONE INDEPENDENT VARIABLE

101. Definition. A *maximum* value of a function is a value *greater* than those immediately preceding or immediately following.

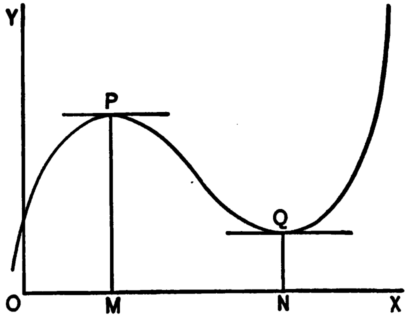
A *minimum* value of a function is a value *less* than those immediately preceding or immediately following.

If the function is represented by the curve $y = f(x)$, then PM represents a maximum value of y or of $f(x)$, and QN represents a minimum value.

102. Conditions for a Maximum or a Minimum.

It is evident that at both P and Q the tangent is parallel to OX, and therefore we have for both maxima and minima,

$$\frac{dy}{dx} = 0.$$



Moreover, as we move along the curve from left to right, at P the slope changes from positive to negative; but at Q, from negative to positive.

In other words,

At P the slope *decreases* as x increases. . . . . (*a*)

At Q the slope *increases* as x increases. . . . . (*b*)

By Art. 21 we have the case (*a*),

when $$\frac{d}{dx}(\text{slope}) < 0. \quad \dots\dots\dots \quad (1)$$

But
$$\frac{d}{dx}(\text{slope}) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

and (1) becomes
$$\frac{d^2y}{dx^2} < 0.$$

Hence when
$$\frac{dy}{dx} = 0, \text{ and } \frac{d^2y}{dx^2} < 0, \quad . \; . \quad . \; . \; . \; . \quad (2)$$
there is a *maximum* value of y.

By similiar reasoning we have the case (*b*),

when
$$\frac{d^2y}{dx^2} > 0.$$

Hence when
$$\frac{dy}{dx} = 0, \text{ and } \frac{d^2y}{dx^2} > 0, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$
there is a *minimum* value of y.

For example, let us find the maximum and minimum values of the function
$$\frac{x^3}{3} - 2x^2 + 3x + 1.$$

Put
$$y = \frac{x^3}{3} - 2x^2 + 3x + 1.$$

Then
$$\frac{dy}{dx} = x^2 - 4x + 3, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

$$\frac{d^2y}{dx^2} = 2x - 4. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

Putting (4) = 0,
$$x^2 - 4x + 3 = 0,$$

whence
$$x = 1 \text{ or } 3.$$

Substituting those values of x in (5), we find

when $x = 1$,
$$\frac{d^2y}{dx^2} = -2 < 0;$$

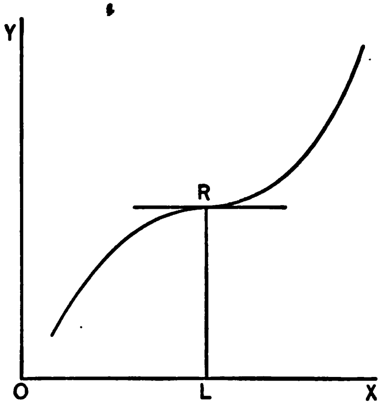
when $x = 3$,
$$\frac{d^2y}{dx^2} = 2 > 0.$$

Hence by (2) and (3),

when $x=1$, y has a maximum value;
when $x=3$, y has a minimum value.

From the given function we find
that the maximum value of y is $y=2\frac{1}{3}$,
and the minimum value of y, $y=1$.

103. In exceptional cases it may happen that a value of x given by $\frac{dy}{dx}=0$, makes $\frac{d^2y}{dx^2}=0$, so that neither (2) nor (3), Art. 102, is satisfied. This would be the case for a point of inflection R (see Art. 158) whose tangent is parallel to OX. Here the ordinate RL is neither a maximum nor a minimum.



But there may be a maximum or minimum value of y, even when $\frac{d^2y}{dx^2}=0$. This is more fully considered in Art. 106. The following article is also applicable to such cases.

104. Second Method of determining Maxima and Minima. Maxima and minima may be determined from the first derivative $\frac{dy}{dx}$ alone, without using $\frac{d^2y}{dx^2}$.

We have seen in Art. 102 that when y is a maximum, as at P, the slope, that is, $\frac{dy}{dx}$, changes from + to −; and when y is a minimum, as at Q, $\frac{dy}{dx}$ changes from − to +. (It is understood that we pass along the curve from left to right.)

By examining the form of $\frac{dy}{dx}$, which should be expressed in factor form, we may determine whether it changes from + to −, or from − to +, for any assigned value of x.

Let us apply this method to the example in Art. 102,

$$\frac{dy}{dx} = x^2 - 4x + 3 = (x-1)(x-3).$$

Here $\frac{dy}{dx}$ can change sign only when $x = 1$ or $x = 3$.

By supposing x to change from a value slightly less, to one slightly greater than 1, we find that $(x - 1)$ changes from − to +; but since the factor $(x - 3)$ is then negative, it follows that $\frac{dy}{dx}$ changes from + to −, when $x = 1$, and denotes a maximum. In the same way, we find that $\frac{dy}{dx}$ changes from − to +, when $x = 3$, and denotes a minimum.

Again, consider the function $y = (x - 4)^5(x + 2)^4$.

Differentiating and writing the result in factor form,

$$\frac{dy}{dx} = 3(3x - 2)(x - 4)^4(x + 2)^3.$$

When $x = \frac{2}{3}$, $\frac{dy}{dx}$ changes from − to +.

When $x = -2$, $\frac{dy}{dx}$ changes from + to −.

When $x = 4$, $\frac{dy}{dx}$ does *not* change sign,

since $(x - 4)^4$ cannot be negative.

Hence we conclude that y is a minimum when $x = \frac{2}{3}$; a maximum when $x = -2$; but neither a maximum nor minimum when $x = 4$.

As this method does not require $\frac{d^2y}{dx^2}$, it is preferable to that of Art. 102, when the second differentiation of y involves much work.

EXAMPLES

1. Find the maximum value of $32x - x^4$. *Ans.* 48.

Find the maximum and minimum values of the following functions.

2. $2x^3 - 3x^2 - 12x + 12$. *Ans.* $x = -1$, gives a maximum 19.
$x = 2$, gives a minimum -8.

3. $2x^3 - 11x^2 + 12x + 10$. *Ans.* $x = \frac{2}{3}$, gives a maximum $13\frac{19}{27}$.
$x = 3$, gives a minimum 1.

4. $x^3 + 9(a - x)^3$. *Ans.* $x = \frac{3a}{2}$, gives a maximum $\frac{9a^3}{4}$.
$x = \frac{3a}{4}$, gives a minimum $\frac{9a^3}{16}$.

5. $(x - 1)(x - 2)(x - 3)$.
Ans. $x = 2 - \frac{1}{\sqrt{3}}$, gives a maximum $\frac{2}{3\sqrt{3}}$.
$x = 2 + \frac{1}{\sqrt{3}}$, gives a minimum $-\frac{2}{3\sqrt{3}}$.

6. $2(3x + 2)^2 - 3x^4$. *Ans.* $x = 2$, gives a maximum 80.

7. Show that $\frac{x^4 - 4x^3 + 8x - 8}{x - 1}$ has no maximum nor minimum.

8. $\frac{a^2}{x} + \frac{b^2}{c - x}$, where a, b, c, are positive.
Ans. $x = \frac{ac}{a + b}$, gives a minimum $\frac{(a + b)^2}{c}$.
$x = \frac{ac}{a - b}$, gives a maximum $\frac{(a - b)^2}{c}$.

9. Show that the greatest value of $\frac{\log x}{x^n}$ is $\frac{1}{ne}$.

10. Show that the greatest value of $\cos 2\theta + \sin\theta$ is $\frac{9}{8}$.

11. Show that the maximum and minimum values of

$$\sin^2\theta + \sin^2\left(\frac{\pi}{3} - \theta\right) \text{ are } \frac{3}{2} \text{ and } \frac{1}{2}.$$

12. Find the maximum value of $a \sin x + b \cos x$. *Ans.* $\sqrt{a^2 + b^2}$.

13. Find the maximum value of $\tan^{-1} x - \tan^{-1} \frac{x}{4}$, the angles being taken in the first quadrant.
Ans. $\tan^{-1} \frac{3}{4}$.

14. Show that the least value of $a^2 \tan^2\theta + b^2 \cot^2\theta$ is the same as that of $a^2 e^{nx} + b^2 e^{-nx}$, and equal to $2ab$.

15. $y = \frac{(a-x)^3}{a-2x}$. *Ans.* A minimum when $x = \frac{a}{4}$.

16. $y = (x-1)^4 (x+2)^3$.

Ans. A maximum when $x = -\frac{5}{7}$; a minimum when $x = 1$; neither when $x = -2$.

17. $y = (x-2)^5 (2x+1)^4$.

Ans. A maximum when $x = -\frac{1}{2}$; a minimum when $x = \frac{11}{18}$, neither when $x = 2$.

105. Case where $\frac{dy}{dx} = \infty$. It is to be noticed that $\frac{dy}{dx}$ may change sign by passing through infinity instead of zero.

Hence if

$$\frac{dy}{dx} = \infty,$$

for a finite value of x, this value should be examined, as well as those given by

$$\frac{dy}{dx} = 0.$$

For example, suppose

$$y = a - b(x-c)^{\frac{2}{3}}.$$

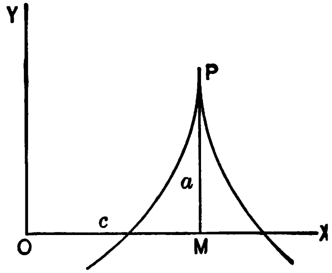
Then
$$\frac{dy}{dx} = -\frac{2b}{3(x-c)^{\frac{1}{3}}};$$

hence we have

$$\frac{dy}{dx} = \infty, \quad \text{when } x = c.$$

It is evident that when $x = c$, $\frac{dy}{dx}$ changes from + to −, indicating a maximum value of y, which is a.

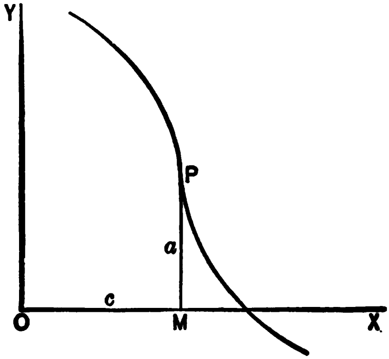
The figure shows the maximum ordinate PM, corresponding to a cusp at P.



On the other hand, suppose $y = a - b(x-c)^{\frac{1}{3}}$.

Then
$$\frac{dy}{dx} = -\frac{b}{3(x-c)^{\frac{2}{3}}} = \infty \quad \text{when } x = c.$$

But as $\frac{dy}{dx}$ does not change sign when $x = c$, there is no maximum nor minimum. The corresponding curve is shown in the figure.



EXAMPLES

Find the maximum and minimum values of the two following functions:

1. $y=(x+1)^{\frac{2}{3}}(x-5)^2$.

Ans. A minimum when $x=5$; a maximum when $x=\frac{1}{2}$;
a minimum when $x=-1$.

2. $y=(2x-a)^{\frac{1}{3}}(x-a)^{\frac{2}{3}}$.

Ans. A maximum when $x=\frac{2a}{3}$; a minimum when $x=a$.

106. Conditions for Maxima and Minima by Taylor's Theorem. Suppose the function $f(x)$ to be a maximum when $x=a$. Then, by the definition in Art. 101,

$$f(a)>f(a+h),$$

and also

$$f(a)>f(a-h),$$

where h is any small but finite quantity. Now, by the substitution of a for x in Taylor's Theorem, we have

$$f(a+h)-f(a)= \quad hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\frac{h^3}{\underline{|3}}f'''(a)+\cdots. \quad (1)^*$$

$$f(a-h)-f(a)=-hf'(a)+\frac{h^2}{\underline{|2}}f''(a)-\frac{h^3}{\underline{|3}}f'''(a)+\cdots. \quad (2)$$

By the hypothesis $\quad f(a+h)-f(a)<0,$

and also $\quad f(a-h)-f(a)<0.$

Hence the second members of both (1) and (2) must be negative.

---

* The rigorous form of Art. 93 may be used here without any change in the context.

By taking h sufficiently small, the first term can be made numerically greater than the sum of all the others, involving h^2, h^3, etc. Thus the sign of the entire second member will be that of the first term. As these have *different* signs in (1) and (2), the second members cannot both be negative unless

$$f'(a)=0.$$

Equations (1) and (2) then become

$$f(a+h)-f(a)=\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+\cdots,$$

$$f(a-h)-f(a)=\frac{h^2}{\lfloor 2}f''(a)-\frac{h^3}{\lfloor 3}f'''(a)+\cdots.$$

The term containing h^2 now determines the sign of the second members. That these may be negative, we must have

$$f''(a)<0.$$

If then $\quad f'(a)=0 \quad \text{and} \quad f''(a)<0,$

$f(a)$ is a maximum.

Similarly, it may be shown that if

$$f'(a)=0 \quad \text{and} \quad f''(a)>0,$$

$f(a)$ will be a minimum.

If $\quad f'(a)=0 \quad \text{and} \quad f''(a)=0,$

similar reasoning will show that for a maximum we must also have

$$f'''(a)=0 \quad \text{and} \quad f^{\text{iv}}(a)<0;$$

and for a minimum

$$f'''(a)=0 \quad \text{and} \quad f^{\text{iv}}(a)>0.$$

The conditions may be generalized as follows:

Suppose that

$$f'(a)=0,\quad f''(a)=0,\quad f'''(a)=0,\quad \cdots \quad f^{n}(a)=0,$$

and that $f^{n+1}(a)$ is not zero.

Then if n is even, $f(a)$ is neither a maximum nor a minimum.

If n is odd, $f(a)$ will be a maximum or a minimum, according as

$$f^{n+1}(a)<0 \quad \text{or} \quad >0.$$

PROBLEMS IN MAXIMA AND MINIMA

1. Divide 10 into two such parts that the product of the square of one and the cube of the other may be the greatest possible.

Let x and $10-x$ be the parts. Then $x^2(10-x)^3$ is to be a maximum. Letting $u = x^2(10-x)^3$, we find

$$\frac{du}{dx} = 5\,x(4-x)(10-x)^2 = 0,$$

from which we find that u is a maximum when $x = 4$. Hence the required parts are 4 and 6.

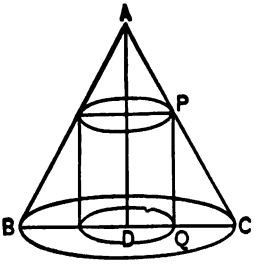
2. A square piece of pasteboard whose side is a has a small square cut out at each corner. Find the side of this square that the remainder may form a box of maximum contents.

Let $x =$ the side of the small square. Then the contents of the box will be $(a-2\,x)^2 x$. Representing this by u, we find that u is a maximum when $x = \frac{a}{6}$, which is the required answer.

3. Find the greatest right cylinder that can be inscribed in a given right cone.

Let $AD = a$, $DC = b$.

Let $x = DQ$, the radius of the base of the cylinder, and $y = PQ$, its altitude.



From the similar triangles ADC, PQC, we find

$$\frac{y}{b-x} = \frac{a}{b}, \qquad y = \frac{a}{b}(b-x).$$

The volume of the cylinder is

$$\pi x^2 y = \pi \frac{a}{b} x^2 (b-x).$$

This will be a maximum when $u = bx^2 - x^3$ is a maximum.

This is found to be when $x = \frac{2}{3}\,b$, the radius of the base of the required cylinder.

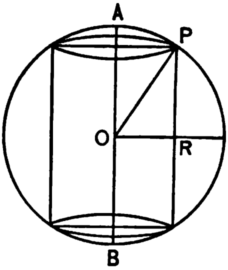
From this, $y = \frac{a}{3}$, the altitude of the cylinder.

4. Determine the right cylinder of the greatest convex surface that can be inscribed in a given sphere.

Let $r = OP$, the radius of the sphere; $x = OR$, the radius of base of cylinder; and $y = PR$, one half its altitude.

From the right triangle OPR we have

$$x^2 + y^2 = r^2.$$



The convex surface of the cylinder is

$$2\pi x \cdot 2y = 4\pi x\sqrt{r^2 - x^2}.$$

We may put u equal to this expression, and determine the value of x that gives a maximum value of u. But the work may be shortened by the following considerations:

$$4\pi x\sqrt{r^2 - x^2} \text{ is a maximum,}$$

when $\qquad x\sqrt{r^2 - x^2}$ is a maximum;

and $\qquad x\sqrt{r^2 - x^2}$ is a maximum,

when its square $\qquad r^2x^2 - x^4$ is a maximum.*

Hence we may put $u = r^2x^2 - x^4$, from which we find u is a maximum, when $x = \frac{r}{\sqrt{2}}$.

From this $\qquad y = \frac{r}{\sqrt{2}}$, giving for the altitude of the cylinder,

$$2y = r\sqrt{2}.$$

Another Method. The equations

$$\text{The convex surface } = 4\pi xy, \quad u = xy, \qquad (1)$$

$$x^2 + y^2 = r^2, \qquad (2)$$

may be used without substituting in (1) the value of y from (2).

---

* Since we are only concerned with the *positive* root of $\sqrt{r^2 - x^2}$.

Differentiating (1),
$$\frac{du}{dx}=y+x\frac{dy}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and differentiating (2), $x+y\dfrac{dy}{dx}=0,\ \dfrac{dy}{dx}=-\dfrac{x}{y}.$

Substituting in (3),
$$\frac{du}{dx}=y-\frac{x^2}{y}=\frac{y^2-x^2}{y}.$$

Since x and y are positive quantities, it is evident that when $x=y$, $\dfrac{du}{dx}$ changes from + to −, giving a maximum value of u.

Combining $x=y$, with (2), we have

$$x=\frac{r}{\sqrt{2}},\ y=\frac{r}{\sqrt{2}}, \text{ as before.}$$

In some problems this method has some advantages over the first.

5. Divide 48 into two parts, such that the sum of the square of one and the cube of the other may be a minimum. *Ans.* $42\frac{5}{8}$, $5\frac{3}{8}$.

6. Divide 20 into two parts, such that the sum of four times the reciprocal of one and nine times the reciprocal of the other may be a minimum. *Ans.* 8, 12.

7. A rectangular sheet of tin 15 inches long and 8 inches wide has a square cut out at each corner. Find the side of this square so that the remainder may form a box of maximum contents.

Ans. $1\frac{2}{3}$ in.

8. How far from the wall of a house must a man, whose eye is 5 feet from the ground, stand, so that a window 5 feet high, whose sill is 9 feet from the ground, may subtend the greatest angle?

Ans. 6 ft.

9. A wall 27 feet high is 8 feet from the side of a house. What is the length of the shortest ladder from the ground over the wall to the house? *Ans.* $13\sqrt{13}=46.87$ ft.

10. A person being in a boat 5 miles from the nearest point of the beach, wishes to reach in the shortest time a place 5 miles from that point along the shore; supposing he can run 6 miles an hour, but row only at the rate of 4 miles an hour, required the place he must land. *Ans.* 929.1 yards from the place to be reached.

11. Find the maximum rectangle that can be inscribed in an ellipse whose semiaxes are a and b.

Ans. The sides are $a\sqrt{2}$ and $b\sqrt{2}$; the area, $2ab$.

12. A rectangular box, open at the top, with a square base, is to be constructed to contain 500 cubic inches. What must be its dimensions to require the least material?

Ans. Altitude, 5 in; side of base, 10 in.

13. A cylindrical tin tomato can is to be made which shall have a given capacity. Find what should be the ratio of the height to the diameter of the base that the smallest amount of tin shall be required. *Ans.* Height = diameter.

14. What are the most economical proportions for an open cylindrical water tank, if the cost of the sides per square foot is two thirds the cost of the bottom per square foot?

Ans. Height $= \frac{3}{4}$ diameter.

15. (*a*) Find the altitude of the rectangle of greatest area that can be inscribed in a circle whose radius is r.

Ans. $r\sqrt{2}$; a square.

(*b*) Find the altitude of the right cylinder of greatest volume that can be inscribed in a sphere whose radius is r. *Ans.* $\dfrac{2r}{\sqrt{3}}$.

16. (*a*) Find the altitude of the isosceles triangle of greatest area inscribed in a circle of radius r. *Ans.* $\dfrac{3r}{2}$; equilateral triangle.

(*b*) Find the altitude of the right cone of greatest volume inscribed in a sphere of radius r. *Ans* $\dfrac{4r}{3}$.

17. (*a*) Find the altitude of the isosceles triangle of least area circumscribed about a circle of radius r.

Ans. $3r$; equilateral triangle.

(*b*) Find the altitude of the right cone of least volume circumscribed about a sphere of radius r. *Ans.* $4r$.

18. A right cone of maximum volume is inscribed in a given right cone, the vertex of one being at the center of the base of the other. Show that the altitude of the inscribed cone is one third the altitude of the other.

19. Find the point of the line, $2x+y=16$, such that the sum of the squares of its distances from $(4, 5)$ and $(6, -3)$ may be a minimum. *Ans.* $(7, 2)$.

20. Find the perpendicular distance from the origin to the line $\frac{x}{a}+\frac{y}{b}=1$, by finding the minimum distance. *Ans.* $\frac{ab}{\sqrt{a^2+b^2}}$.

21. A vessel is sailing due north 10 miles per hour. Another vessel 190 miles north of the first is sailing 15 miles per hour on a course East 30° South. When will they be nearest together, and what is their least distance apart?

Ans. In 7 hrs. Distance $15\sqrt{57}=113.25$ mi.

22. A vessel is anchored 3 miles off the shore. Opposite a point 3 miles farther along the shore, another vessel is anchored 9 miles from the shore. A boat from the first vessel is to land a passenger on the shore and then proceed to the other vessel. What is the shortest course of the boat? *Ans.* 13 mi.

23. The velocity of waves of length λ in deep water is proportional to $\sqrt{\frac{\lambda}{a}+\frac{a}{\lambda}}$, where a is a certain linear magnitude. Show that the velocity is a minimum when $\lambda=a$.

24. Assuming that the current in a voltaic cell is $C=\frac{E}{r+R}$, E being the electromotive force, r the internal, and R the external, resistance; and that the power given out is $P=RC^2$; show that P is a maximum when $R=r$.

25. From a given circular sheet of metal, to cut out a sector, so that the remainder may form a conical vessel of maximum capacity.

Ans. Angle of sector $= \left(1 - \frac{\sqrt{6}}{3}\right) 2\pi = 66^\circ\ 4'$.

26. Find the height of a light on a wall so as best to illuminate a point on the floor a feet from the wall; assuming that the illumination is inversely as the square of the distance from the light, and directly as the sine of the inclination of the rays to the floor.

Ans. $\frac{a}{\sqrt{2}}$.

27. At what point on the line joining the centres of two spheres must a light be placed, to illuminate the largest amount of spherical surface?

Ans. The centres being A, B; the radii, a, b; and P the required point; $\overline{AP}^2 : \overline{PB}^2 = a^3 : b^3$.

28. (*a*) The strength of a rectangular beam varies as the breadth and the square of the depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log whose diameter is $2a$.

Ans. Breadth $= \frac{2a}{\sqrt{3}}$. Depth $= 2a\sqrt{\frac{2}{3}}$.

(*b*) The stiffness of a rectangular beam varies as the breadth and the cube of the depth. Find the dimensions of the stiffest beam that can be cut from the log. *Ans.* Breadth $= a$. Depth $= a\sqrt{3}$.

29. The work of propelling a steamer through the water varies as the cube of her speed. Find the most economical speed against a current running 4 miles per hour. *Ans.* 6 mi. per hr.

30. The cost of fuel consumed in propelling a steamer through the water varies as the cube of her speed, and is $25 per hour when the speed is 10 miles per hour. The other expenses are $100 per hour. Find the most economical speed.

Ans. $\sqrt[3]{2000} = 12.6$ mi. per hr.

31. A weight of 1000 lbs. hanging 2 feet from one end of a lever is to be raised by an upward force at the other end. Supposing the lever to weigh 10 lbs. per foot, find its length that the force may be a minimum. *Ans.* 20 ft.

32. (*a*) The lower corner of a leaf, whose width is a, is folded over so as just to reach the inner edge of the page. Find the width of the part folded over, when the length of the crease is a minimum.
Ans. $\frac{3}{4}a$.

(*b*) In the preceding example, find when the area of the triangle folded over is a minimum. *Ans.* When the width folded is $\frac{2}{3}a$.

33. A steel girder 25 feet long is moved on rollers along a passageway 12.8 feet wide, and into a corridor at right angles to the passageway. Neglecting the horizontal width of the girder, how wide must the corridor be, in order that the girder may go around the corner? *Ans.* 5.4 ft.

34. Find the altitude of the least isosceles triangle that can be circumscribed about an ellipse whose semiaxes are a and b, the base of the triangle being parallel to the major axis. *Ans.* $3b$.

35. A tangent is drawn to the ellipse whose semiaxes are a and b, such that the part intercepted by the axes is a minimum. Show that its length is $a + b$.

CHAPTER XI

PARTIAL DIFFERENTIATION

107. Functions of Two or More Independent Variables. In the preceding chapters differentiation has been applied only to functions of one independent variable. We shall now consider functions of more than one variable.

Let $$u = f(x, y)$$

be a function of the two independent variables x and y.

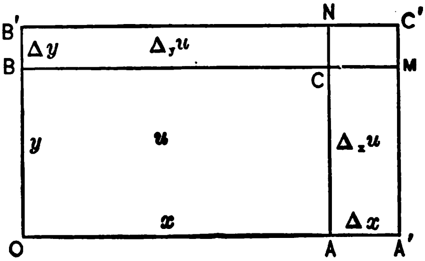
Since x and y are independent of each other, we may suppose x to vary while y remains constant, or y to vary while x remains constant; or we may suppose x and y to vary simultaneously. We must distinguish between the changes in u resulting from these different suppositions.

Let $\Delta_x u$ denote the increment in u resulting from a change in x only, and $\Delta_y u$ the increment in u from a change in y only.

Let Δu, called the *total increment of* u, be the increment when x and y both change.

Suppose u the area of a rectangle whose sides are x and y.

Then $u = xy$.



If x changes from OA to OA', while y remains constant, u is increased by the rectangle AM.

That is, $$\Delta_x u = \text{area } AM.$$

If y changes from OB to OB', while x remains constant, u is increased by the rectangle BN.

That is, $$\Delta_y u = \text{area } BN.$$

If x and y both change together, we have for the total increment of u, $$\Delta u = \text{area } AM + \text{area } BN + \text{area } MN.$$

108. Partial Differentiation. This supposes only one of the independent variables to vary at the same time, so that the differentiation is performed by the same rules that have been applied to functions of a single variable.

If we differentiate $u = f(x, y)$, supposing x to vary, y remaining constant, we obtain $\frac{du}{dx}$.

If we differentiate, supposing y to vary, x remaining constant, we obtain $\frac{du}{dy}$.

The derivatives, $\frac{du}{dx}$, $\frac{du}{dy}$, thus derived, are called *partial derivatives*, and a special notation, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, is used for them.

For example, if $$u = x^3 + 2x^2y - y^3,$$

$$\frac{\partial u}{\partial x} = 3x^2 + 4xy, \quad \text{the } x\text{-derivative of } u.$$

$$\frac{\partial u}{\partial y} = 2x^2 - 3y^2, \quad \text{the } y\text{-derivative of } u.$$

In general, whatever the number of independent variables, the partial derivatives are obtained by supposing only one to vary at a time.

EXAMPLES

Derive by partial differentiation the following results:

1. $u = \frac{xy}{x+y}$, $\qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$

2. $z = (ax^2 + 2bxy + cy^2)^n$, $\qquad (bx + cy)\frac{\partial z}{\partial x} = (ax + by)\frac{\partial z}{\partial y}$

3. $u=(y-z)(z-x)(x-y)$, $\qquad \dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0.$

4. $r=\sqrt{x^2+y^2}$, $\qquad y\dfrac{\partial r}{\partial x}=x\dfrac{\partial r}{\partial y}.$

5. $u=\log(x^3+ax^2y+bxy^2+cy^3)$, $\qquad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3.$

6. $u=\dfrac{y}{z}+\dfrac{z}{x}+\dfrac{x}{y}$, $\qquad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=0.$

7. $u=\dfrac{e^{xy}}{e^x+e^y}$, $\qquad \dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=(x+y-1)\,u.$

8. $u=\log\dfrac{x-y}{x+y}+2\tan^{-1}\dfrac{x}{y}$, $\qquad y\dfrac{\partial u}{\partial x}+x\dfrac{\partial u}{\partial y}+\dfrac{4x^2}{x^2+y^2}=0.$

9. $z=(x+y)(x^2-y^2)^n$, $\qquad y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial y}=z.$

10. $u=\dfrac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$, $\qquad \left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2+\left(\dfrac{\partial u}{\partial z}\right)^2=\dfrac{1}{(x^2+y^2+z^2)^2}$

11. $z=\log_y x+\log_x y$, $\qquad x\log x\dfrac{\partial z}{\partial x}+y\log y\dfrac{\partial z}{\partial y}=0.$

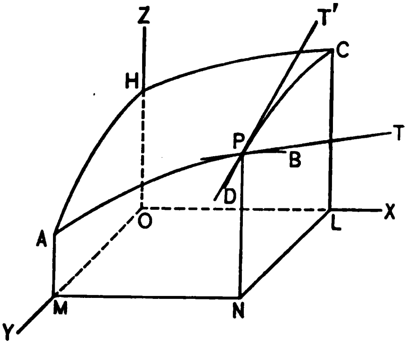
12. $u=e^x\sin y+e^y\sin x$,

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{2x}+e^{2y}+2\,e^{x+y}\sin(x+y).$$

13. $u=\log(\tan x+\tan y+\tan z)$,

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2.$$

109. Geometrical Illustration of Partial Derivatives. Let $z=f(x, y)$ be the equation of the surface *APCH*.



The ordinate *PN* is thus given for every point *N* in the plane *XY*.

Let *APB* and *CPD* be sections of the surface by planes through *P*, parallel to *XZ* and *YZ*, respectively.

If x and y both vary, *P* moves to some other position on the surface.

If x varies, y remaining constant, *P* moves on the curve of intersection *APB*.

Hence $\frac{\partial z}{\partial x}$ is the slope of the curve *APB* at *P*.

If y varies, x remaining constant, *P* moves on the curve *CPD*.

Hence $\frac{\partial z}{\partial y}$ is the slope of the curve *CPD* at *P*.

110. Equation of Tangent Plane. Angles with Coördinate Planes. In the figure of the preceding article, let *P* be the point (x', y', z'); *PT*, the tangent to *APB* in the plane *APNM*; and *PT'*, the tangent to *CPD* in the plane *CPNL*.

It is evident from the preceding article that the equations of *PT* are

$$z-z'=\frac{\partial z'}{\partial x'}(x-x'), \quad y=y', \quad . \quad . \qquad . \quad . \quad . \quad (1)^*$$

and of *PT'*,

$$z-z'=\frac{\partial z'}{\partial y'}(y-y'), \quad x=x'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

* $\frac{\partial z'}{\partial x'}$, $\frac{\partial z'}{\partial y'}$ denote the values of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, respectively, for (x', y', z').

The plane tangent to the surface at P contains the tangent lines PT and PT'. Its equation is

$$z-z'=\frac{\partial z'}{\partial x'}(x-x')+\frac{\partial z'}{\partial y'}(y-y'). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For (3) is of the first degree with respect to the current variables x, y, z, and is satisfied by (1), and also by (2).

The equations of the normal through P are those of a line through (x', y', z') perpendicular to (3). Its equations are

$$\frac{x-x'}{\frac{\partial z'}{\partial x'}}=\frac{y-y'}{\frac{\partial z'}{\partial y'}}=-(z-z'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The angles made by the tangent plane with the coördinate planes are equal to the inclinations of the normal to the coördinate axes.

By analytic geometry of three dimensions, the direction cosines of the line perpendicular to (3) are proportional to

$$\frac{\partial z'}{\partial x'},\ \frac{\partial z'}{\partial y'},\ -1.$$

Hence, if α, β, γ, are the inclinations of the normal to OX, OY, OZ, respectively,

$$\frac{\cos\alpha}{\frac{\partial z'}{\partial x'}}=\frac{\cos\beta}{\frac{\partial z'}{\partial y'}}=\frac{\cos\gamma}{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Also $$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From (5) and (6) we find, dropping the accents,

$$\cos^2\alpha=\frac{\left(\frac{\partial z}{\partial x}\right)^2}{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2},$$

$$\cos^2\beta=\frac{\left(\frac{\partial z}{\partial y}\right)^2}{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2},$$

$$\cos^2\gamma=\frac{1}{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}. \quad . \quad . \quad . \quad . \qquad (7)$$

For the inclination of the tangent plane to XY, we have from (7),

$$\sec^2\gamma=1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and
$$\tan^2\gamma=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

The term *slope* used in geometry of two dimensions may thus be extended to three dimensions, as the tangent of the angle made by the tangent plane with the plane XY. In this sense,

$$\text{the slope}=\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}.$$

EXAMPLES

1. Find the equations of the tangent plane and normal, to the sphere $x^2+y^2+z^2=a^2$, at $(x', y,' z')$.

$$\frac{\partial z}{\partial x}=-\frac{x}{z},\quad \frac{\partial z}{\partial y}=-\frac{y}{z};$$

Hence
$$\frac{\partial z'}{\partial x'}=-\frac{x'}{z'},\quad \frac{\partial z'}{\partial y'}=-\frac{y'}{z'}.$$

Substituting in (3), $z-z'=-\frac{x'}{z'}(x-x')-\frac{y'}{z'}(y-y'),$

$$xx'+yy'+zz'=x'^2+y'^2+z'^2=a^2. \quad \textit{Ans.}$$

From (4) we find for the normal

$$(x - x')\frac{z'}{x'} = (y - y')\frac{z'}{y'} = z - z',$$

$$\frac{x}{x'} - 1 = \frac{y}{y'} - 1 = \frac{z}{z'} - 1 \qquad \frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}. \quad \textit{Ans.}$$

2. Find the equations of tangent plane and normal to the cone,

$$3x^2 - y^2 + 2z^2 = 0, \text{ at } (x', y', z').$$

Ans. $3xx' - yy' + 2zz' = 0, \qquad \dfrac{x - x'}{3x'} = \dfrac{y - y'}{-y'} = \dfrac{z - z'}{2z'}.$

3. Find the equation of the tangent plane to the elliptic paraboloid, $z = 3x^2 + 2y^2$, at the point (1, 2, 11).

Ans. $6x + 8y - z = 11.$

4. Find the equations of tangent plane and normal to the ellipsoid,

$$x^2 + 2y^2 + 3z^2 = 20,$$

at the point $x = 3$, $y = 2$, z being positive.

Ans. $3x + 4y + 3z = 20; \ x = z + 2, \ 3y = 4z + 2$

Find the *slope* of this tangent plane. *Ans.* $\frac{5}{3}$

5. Find the equation of the tangent plane to the sphere,

$$x^2 + y^2 + z^2 - 2x + 2y = 1, \text{ at } (x', y', z').$$

Ans. $xx' + yy' + zz' - x - x' + y + y' = 1.$

111. Partial Derivatives of Higher Orders. By successive differentiation, the independent variables varying only one at a time, we may obtain

$$\frac{\partial^2 u}{\partial x^2}, \ \frac{\partial^2 u}{\partial y^2}, \ \frac{\partial^3 u}{\partial x^3}, \ \frac{\partial^4 u}{\partial y^4}, \cdots.$$

If we differentiate u with respect to x, then this result with respect to y, we obtain $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$, which is written $\dfrac{\partial^2 u}{\partial y\,\partial x}$.

Similarly, $\frac{\partial^3 u}{\partial y\,\partial x^2}$ is the result of three successive differentiations, two with respect to x, and one with respect to y. It will now be shown that this result is independent of the order of these differentiations. In other words, the operations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are commutative.

That is, $\frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}$, $\frac{\partial^3 u}{\partial y\,\partial x^2}=\frac{\partial^3 u}{\partial x\,\partial y\,\partial x}=\frac{\partial^3 u}{\partial x^2\,\partial y}$.

112. Given $$u=f(x,y), \quad \ldots \quad (1)$$

to prove that $$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right).$$

Supposing x to change in (1), y being constant,

$$\frac{\Delta u}{\Delta x}=\frac{f(x+\Delta x, y)-f(x,y)}{\Delta x}. \quad \ldots \quad (2)$$

Now supposing y to change in (2), x being constant,

$$\frac{\Delta}{\Delta y}\left(\frac{\Delta u}{\Delta x}\right)=\frac{f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)-f(x+\Delta x, y)+f(x,y)}{\Delta y\Delta x}.$$

Reversing the above order, we find

$$\frac{\Delta u}{\Delta y}=\frac{f(x, y+\Delta y)-f(x,y)}{\Delta y}, \quad \text{and}$$

$$\frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta y}\right)=\frac{f(x+\Delta x, y+\Delta y)-f(x+\Delta x, y)-f(x, y+\Delta y)+f(x,y)}{\Delta x\Delta y}.$$

Hence $$\frac{\Delta}{\Delta y}\left(\frac{\Delta u}{\Delta x}\right)=\frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta y}\right). \quad \ldots \quad (3)$$

The mean value theorem, (2), Art. 90, may be expressed in the form $\frac{\Delta u}{\Delta x}=f'(x+\theta\cdot\Delta x)$, where $u=f(x)$. $0<\theta<1$.

In the present case, where $u = f(x, y)$,

$$\frac{\Delta u}{\Delta x} = f_x(x + \theta_1 \cdot \Delta x, y).^*$$

$$\frac{\Delta}{\Delta y}\left(\frac{\Delta u}{\Delta x}\right) = \frac{\Delta}{\Delta y} f_x(x + \theta_1 \cdot \Delta x, y) = f_{yx}(x + \theta_1 \cdot \Delta x, y + \theta_2 \cdot \Delta y).$$

Similarly, $\dfrac{\Delta u}{\Delta y} = f_y(x, y + \theta_3 \cdot \Delta y)$,

and
$$\frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta y}\right) = f_{xy}(x + \theta_4 \cdot \Delta x, y + \theta_3 \cdot \Delta y).$$

By (3) $\quad f_{yx}(x + \theta_1 \cdot \Delta x, y + \theta_2 \cdot \Delta y) = f_{xy}(x + \theta_4 \cdot \Delta x, y + \theta_3 \cdot \Delta y).$

Taking the limits as Δx, Δy, approach zero, and assuming the functions involved to be continuous,

$$f_{yx}(x, y) = f_{xy}(x, y).$$

That is, $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$, or $\dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial^2 u}{\partial x\, \partial y}$.

This principle, *that the order of differentiation is immaterial,* may be extended to any number of differentiations.

Thus,
$$\frac{\partial^3 u}{\partial y\, \partial x^2} = \frac{\partial^2}{\partial y\, \partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2}{\partial x\, \partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^3 u}{\partial x\, \partial y\, \partial x}$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\, \partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x\, \partial y}\right) = \frac{\partial^3 u}{\partial x^2\, \partial y}.$$

It is evident that the same is true of functions of three or more variables.

---

* $f_x(x, y) = \dfrac{\partial}{\partial x} f(x, y)$, $\quad f_y(x, y) = \dfrac{\partial}{\partial y} f(x, y)$,

$f_{yx}(x, y) = \dfrac{\partial}{\partial y}\left[\dfrac{\partial}{\partial x} f(x, y)\right]$, $\quad f_{xy}(x, y) = \dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial y} f(x, y)\right]$.

EXAMPLES

Verify $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$ in Exs. 1–3.

1. $u=\dfrac{ax+by}{ay+bx}$, **2.** $u=xy\log\dfrac{x}{y}$. **3.** $u=(x+y)\,e^{x-y}$.

Derive the following results:

4. $u=ax^4+6\,bx^2y^2+cy^4$, $\dfrac{\partial^4 u}{\partial x^2\,\partial y^2}=\dfrac{\partial^4 u}{\partial x\,\partial y\,\partial x\,\partial y}=\dfrac{\partial^4 u}{\partial x\,\partial y^2\,\partial x}$.

5. $u=\log\,(x^2+y^2)$, $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

6. $z=(3x+y)^3+\sin\,(2x-y)$, find $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\,\partial y}-6\dfrac{\partial^2 z}{\partial y^2}$.

7. $u=\dfrac{x^2}{y}+\log\dfrac{x}{y}$, find $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$

8. $z=x^2\tan^{-1}\dfrac{y}{x}-y^2\tan^{-1}\dfrac{x}{y}$, $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{x^2-y^2}{x^2+y^2}$.

9. $q=(r^n+r^{-n})\cos n\theta$, $\dfrac{\partial^2 q}{\partial r^2}+\dfrac{1}{r}\,\dfrac{\partial q}{\partial r}+\dfrac{1}{r^2}\,\dfrac{\partial^2 q}{\partial \theta^2}=0.$

10. $u=\log\,(e^x+e^y+e^z)$, $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}=2e^{x+y+z-3u}$.

11. $u=z\tan^{-1}\dfrac{x}{y}$, $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0.$

12. $u=\log\,(x^2+y^2+z^2)$, $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2}{x^2+y^2+z^2}$.

13. $u=y^2z^2e^{\frac{x}{2}}+z^2x^2e^{\frac{y}{2}}+x^2y^2e^{\frac{z}{2}}$, $\dfrac{\partial^6 u}{\partial x^2\partial y^2\partial z^2}=e^{\frac{x}{2}}+e^{\frac{y}{2}}+e^{\frac{z}{2}}$.

14. $u=\sin\,(y+z)\sin\,(z+x)\sin\,(x+y)$,

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=2\cos\,(2x+2y+2z).$$

113. Total Derivative. Total Differential. In Art. 107 we have referred to the change in u when x and y vary simultaneously. This change is called the *total increment of u*. Thus the total increment of

$$u = f(x, y)$$

is

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

The terms *total derivative* and *total differential* are also used. For example, let

$$u = x^3y - 3x^2y^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and suppose x and y to be functions of a variable t.

Differentiating with respect to t,

$$\frac{du}{dt} = \frac{d}{dt}(x^3y) - \frac{d}{dt}(3x^2y^3)$$

$$= x^3\frac{dy}{dt} + 3x^2y\frac{dx}{dt} - 6x^2y\frac{dy}{dt} - 6xy^3\frac{dx}{dt}$$

$$= (3x^2y - 6xy^2)\frac{dx}{dt} + (x^3 - 6x^2y)\frac{dy}{dt}. \quad . \quad . \quad . \quad . \quad (2)$$

But from (1) we find

$$\frac{\partial u}{\partial x} = 3x^2y - 6xy^2, \qquad \frac{\partial u}{\partial y} = x^3 - 6x^2y.$$

So that (2) may be written

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we had used *differentials* in differentiating (1) we should have obtained

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$\frac{du}{dt}$ in (2) and (3) is called the *total derivative*, and du in (4) the *total differential*, of u.

We proceed to show that (3) and (4) are true for any function of x and y.

Noticing that Δu is the total increment of u, and $\Delta_x u$, $\Delta_y u$, the partial increments, when x and y vary separately, let

$$u = f(x, y), \qquad x \text{ and } y \text{ being functions of } t.$$

$$u' = f(x + \Delta x, y),$$

$$u'' = f(x + \Delta x, y + \Delta y).$$

Then

$$\Delta_x u = u' - u,$$

$$\Delta_y u' = u'' - u',$$

$$\Delta u = u'' - u.$$

Hence

$$\Delta u = \Delta_x u + \Delta_y u',$$

and

$$\frac{\Delta u}{\Delta t} = \frac{\Delta_x u}{\Delta x}\frac{\Delta x}{\Delta t} + \frac{\Delta_y u'}{\Delta y}\frac{\Delta y}{\Delta t}.$$

Taking the limits of each member, as Δt, and consequently Δx, Δy, approach zero,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}, \quad \dots\dots\dots \quad (5)$$

since the limit of u' is u.

This may be written in the differential form

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy. \quad \dots\dots\dots \quad (6)$$

In the same way, if $u = f(x, y, z)$, where x, y, z, are functions of t, we find

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}, \quad \dots\dots\dots \quad (7)$$

and

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}dz. \quad \dots\dots\dots \quad (8)$$

We may write in (8)

$$\frac{\partial u}{\partial x}\,dx = d_x u, \qquad \frac{\partial u}{\partial y}\,dy = d_y u, \qquad \frac{\partial u}{\partial z}\,dz = d_z u,$$

giving

$$du = d_x u + d_y u + d_z u,$$

that is, the *total differential* of u is the sum of its *partial differentials*.

This principle, as expressed by $du = d_x u + d_y u$, may be illustrated by the figure of Art. 107, from which we have

$$\Delta u = \Delta_x u + \Delta_y u + \text{area } MN,$$

that is,
$$\Delta u = \Delta_x u + \Delta_y u + \Delta x\, \Delta y.$$

As Δx and Δy approach zero, the last term diminishes more rapidly than the others, and we may write

$$\Delta u = \Delta_x u + \Delta_y u, \qquad \text{approximately},$$

the closeness of the approximation increasing as Δx and Δy approach zero.

If in (5) we suppose $t = x$,

then
$$u = f(x, y), \qquad y \text{ being a function of } x;$$

and (5) becomes
$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}. \quad \text{. . . . . (9)}$$

Similarly, if in (7), $t = x$,

$$u = f(x, y, z), \qquad y \text{ and } z \text{ being functions of } x;$$

whence
$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}. \quad \text{. . . . . . (10)}$$

EXAMPLES

Find the total derivative of u by (5) or (7) in the three following:

1. $u = f(x, y, z)$, where $x = t^2$, $y = t^3$, $z = \frac{1}{t}$.

$$\frac{du}{dt} = 2t\frac{\partial u}{\partial x} + 3t^2\frac{\partial u}{\partial y} - \frac{1}{t^2}\frac{\partial u}{\partial z}.$$

2. $u = \log(x^2 - y^2)$, where $x = a\cos t$, $y = a\sin t$.

$$\frac{du}{dt} = -2\tan 2t.$$

3. $u = \tan^{-1}\frac{x}{y}$, where $x = 2t$, $y = 1 - t^2$. $\quad \frac{du}{dt} = \frac{2}{1 + t^2}.$

Apply (10) to the two following:

4. $u = f(x, y, z)$, where $y = x^2 - x, \quad z = x^3 - x^2$.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + (2x - 1)\frac{\partial u}{\partial y} + (3x^2 - 2x)\frac{\partial u}{\partial z}.$$

5. $u = \tan^{-1}\frac{xy}{z}$, where $y = 3 - x^2, \quad z = 1 - 3x^2, \qquad \frac{du}{dx} = \frac{3}{1 + x^2}$

Find the total differential by (6) or (8) in the following:

6. $u = ax^2 + 2bxy + cy^2, \qquad du = 2(ax + by)\,dx + 2(bx + cy)\,dy.$

7. $u = x^{\log y}, \qquad du = u\left(\frac{\log y}{x}dx + \frac{\log x}{y}dy\right).$

8. $u = \log\frac{\sin\frac{1}{2}(x + y)}{\sin\frac{1}{2}(x - y)}, \qquad du = \frac{\sin y\,dx - \sin x\,dy}{\cos x - \cos y}.$

9. $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$

$du = 2(ax + hy + gz)\,dx + 2(hx + by + fz)\,dy + 2(gx + fy + cz)\,dz.$

10. $u = x^{yz}, \qquad du = x^{yz-1}(yz\,dx + zx\log x\,dy + xy\log x\,dz).$

11. $u = \tan^2 x \tan^2 y \tan^2 z, \qquad du = 4u\left(\frac{dx}{\sin 2x} + \frac{dy}{\sin 2y} + \frac{dz}{\sin 2z}\right).$

If the variable t in (5) and (7) denotes the time, we have the relation between the rates of increase of the variables.

For illustration consider the following example:

12. One side of a plane triangle is 8 feet long, and increasing 4 inches per second; another side is 5 feet, and decreasing 2 inches per second. The included angle is 60°, and increasing 2° per second. At what rate is the area of the triangle increasing?

The area $\Delta = \frac{1}{2} bc \sin A$, from which

$$\frac{d\Delta}{dt} = \frac{c}{2}\sin A\frac{db}{dt} + \frac{b}{2}\sin A\frac{dc}{dt} + \frac{bc}{2}\cos A\frac{dA}{dt}$$

$$= \frac{c}{2}\sin A \cdot \frac{1}{3} + \frac{b}{2}\sin A \cdot -\frac{1}{6} + \frac{bc}{2}\cos A\frac{\pi}{90}$$

$$= .4934 \text{ sq. ft.} = 71.05 \text{ sq. in. per sec.}$$

13. One side of a rectangle is 10 inches long, and increasing uniformly 2 inches per second. The other side is 15 inches long, and decreasing uniformly 1 inch per second. At what rate is the area increasing? *Ans.* 20 sq. in. per sec.

At what rate after the lapse of 2 seconds? *Ans.* 12 sq. in. per sec.

14. The altitude of a circular cone is 100 inches, and decreasing 10 inches per second, and the radius of the base is 50 inches and increasing 5 inches per second. At what rate is the volume increasing? *Ans.* 15.15 cu. ft. per sec.

15. In Ex. 12, at what rate is the side opposite the given angle increasing? *Ans.* 4.93 in. per sec.

114. Differentiation of an Implicit Function. (See Art. 66.) The derivative of an implicit function may be expressed in terms of partial derivatives.

The equation connecting y and x, by transposing all the terms to one member, may be represented by

$$\phi(x, y) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $u = \phi(x, y)$.

From (9), Art. 113, we have for the total derivative of u,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

But by (1) x and y must have such values that u may be zero, that is, a constant; and therefore its total derivative $\frac{du}{dx}$ must be zero.

Hence
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0,$$

and
$$\frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\quad \text{. . . . .} \quad (2)$$

For example, find $\frac{dy}{dx}$ from
$$x^3y^2+x^2y^3=a^5.$$

Let
$$u=x^3y^2+x^2y^3-a^5.$$

$$\frac{\partial u}{\partial x}=3x^2y^2+2xy^3,\qquad \frac{\partial u}{\partial y}=2x^3y+3x^2y^2.$$

By (2)
$$\frac{dy}{dx}=-\frac{3x^2y^2+2xy^3}{2x^3y+3x^2y^2}=-\frac{3xy+2y^2}{2x^2+3xy}.$$

In the same way find the first derivatives in the examples of Art. 66.

115. Extension of Taylor's Theorem to Functions of Two Independent Variables. If we apply Taylor's Theorem

to
$$f(x+h,\,y+k),$$

regarding x as the only variable, we have

$$f(x+h,\,y+k)=f(x,\,y+k)+h\frac{\partial}{\partial x}f(x,\,y+k)$$
$$+\frac{h^2}{\underline{\lfloor 2}}\frac{\partial^2}{\partial x^2}f(x,\,y+k)+\cdots.\quad (1)$$

Now expanding $f(x, y+k)$, regarding y as the only variable,

$$f(x, y+k) = f(x, y) + k\frac{\partial}{\partial y}f(x, y) + \frac{k^2}{\underline{|2}}\frac{\partial^2}{\partial y^2}f(x, y) + \cdots.$$

Substituting this in (1),

$$f(x+h, y+k) = f(x, y) + h\frac{\partial}{\partial x}f(x, y) + k\frac{\partial}{\partial y}f(x, y)$$

$$+\frac{1}{\underline{|2}}\left[h^2\frac{\partial^2}{\partial x^2}f(x, y) + 2\,hk\frac{\partial^2}{\partial x\,\partial y}f(x, y) + k^2\frac{\partial^2}{\partial y^2}f(x, y)\right] + \cdots. \quad (2)$$

This may be expressed in the symbolic form thus:

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x, y)$$

$$+\frac{1}{\underline{|2}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \frac{1}{\underline{|3}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \cdots,$$

where $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$ is to be expanded by the Binomial Theorem, as if $h\frac{\partial}{\partial x}$ and $k\frac{\partial}{dy}$ were the two terms of the binomial, and the resulting terms applied separately to $f(x, y)$.

116. Taylor's Theorem applied to Functions of Any Number of Independent Variables. By a method similar to that of the preceding article we shall find

$$f(x+h, y+k, z+l) = f(x, y, z) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)f(x, y, z)$$

$$+\frac{1}{\underline{|2}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(x, y, z) + \cdots.$$

This expansion may be extended to any number of variables.

EXAMPLES

1. Expand $\log(x+h)\log(y+k)$.

Let $u = f(x, y) = \log x \log y$, $\dfrac{\partial u}{\partial x} = \dfrac{\log y}{x}$, $\dfrac{\partial u}{\partial y} = \dfrac{\log x}{y}$,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\log y}{x^2},\quad \frac{\partial^2 u}{\partial y \partial x} = \frac{1}{xy},\quad \frac{\partial^2 u}{\partial y^2} = -\frac{\log x}{y^2}.$$

By (2), Art. 115, $\log(x+h)\log(y+k) = \log x \log y$

$$+\frac{h}{x}\log y + \frac{k}{y}\log x - \frac{h^2}{2x^2}\log y + \frac{hk}{xy} - \frac{k^2}{2y^2}\log x + \cdots.$$

2. $(x+h)^3(y+k)^2 = x^3y^2 + 3hx^2y^2 + 2kx^3y$

$$+ 3h^2xy^2 + 6hkx^2y + k^2x^3 + \cdots$$

3. $\sin[(x+h)(y+k)] = \sin(xy) + hy\cos(xy) + kx\cos(xy)$

$$-\frac{h^2y^2}{2}\sin(xy) + hk[\cos(xy) - xy\sin(xy)] - \frac{k^2x^2}{2}\sin(xy) + \cdots$$

4. $\log(e^{x+h} + e^{y+k}) = \log(e^x + e^y) + \dfrac{he^x + ke^y}{e^x + e^y} + \dfrac{e^xe^y(h-k)^2}{2(e^x+e^y)^2} + \cdots.$

CHAPTER XII

CHANGE OF THE VARIABLES IN DERIVATIVES

117. To express $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ ⋯ in terms of $\frac{dx}{dy}, \frac{d^2x}{dy^2}, \frac{d^3x}{dy^3}$, ⋯. This is *changing the independent variable from x to y.*

By (1), Art. 56,

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}. \quad \dots\dots\dots \quad (1)$$

By (3), Art. 56,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{d}{dy}\frac{dy}{dx}\cdot\frac{dy}{dx}.$$

From (1),

$$\frac{d}{dy}\frac{dy}{dx}=\frac{d}{dy}\frac{1}{\frac{dx}{dy}}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}.$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}. \quad \dots\dots\dots \quad (2)$$

Similarly,

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\frac{d^2y}{dx^2}=\frac{d}{dy}\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}.$$

From (2),

$$\frac{d}{dy}\frac{d^2y}{dx^2}=\frac{3\left(\frac{d^2x}{dy^2}\right)^2-\frac{dx}{dy}\frac{d^3x}{dy^3}}{\left(\frac{dx}{dy}\right)^4}.$$

$$\therefore \frac{d^3y}{dx^3}=\frac{3\left(\frac{d^2x}{dy^2}\right)^2-\frac{dx}{dy}\frac{d^3x}{dy^3}}{\left(\frac{dx}{dy}\right)^5}. \quad \dots\dots\dots \quad (3)$$

It is sometimes necessary in the derivatives,

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots,$$

to introduce a new variable z in place of x or y, z being a given function of the variable removed.

There are two cases, according as z replaces y or x.

118. First. To express $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$, in terms of $\frac{dz}{dx}, \frac{d^2z}{dx^2}, \frac{d^3z}{dx^3}, \ldots$, where y is a given function of z.

By (3), Art. 56,
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + \frac{dy}{dz}\frac{d^2z}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}.$$

Similarly, we find

$$\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3}\left(\frac{dz}{dx}\right)^3 + 3\frac{d^2y}{dz^2}\frac{dz}{dx}\frac{d^2z}{dx^2} + \frac{dy}{dz}\frac{d^3z}{dx^3}.$$

Similarly, $\frac{d^4y}{dx^4}, \frac{d^5y}{dx^5}, \ldots$, may be expressed in terms of z and x.

It is to be noticed that in this case there is no change of the *independent* variable, which remains x.

For example, suppose $y = z^3$.

Then
$$\frac{dy}{dx} = 3z^2\frac{dz}{dx}.$$

$$\frac{d^2y}{dx^2} = 6z\left(\frac{dz}{dx}\right)^2 + 3z^2\frac{d^2z}{dx^2}.$$

$$\frac{d^3y}{dx^3} = 6\left(\frac{dz}{dx}\right)^3 + 18z\frac{dz}{dx}\frac{d^2z}{dx^2} + 3z^2\frac{d^3z}{dx^3}.$$

119. Second. To express $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots,$ in terms of $\frac{dy}{dz}, \frac{d^2y}{dz^2}, \frac{d^3y}{dz^3}, \dots,$ where x is a given function of z.

This is *changing the independent variable from x to z.*

By (3), Art. 56,
$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{\frac{dy}{dz}}{\frac{dx}{dz}}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx}=\frac{\frac{d}{dz}\left(\frac{dy}{dx}\right)}{\frac{dx}{dz}}$$

$$=\frac{\frac{dx}{dz}\frac{d^2y}{dz^2}-\frac{dy}{dz}\frac{d^2x}{dz^2}}{\left(\frac{dx}{dz}\right)^3}.$$

Similarly, higher derivatives may be expressed. In practice it is generally easier to work out each case by itself.

For example, suppose $x=z^3$.

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}.$$

But
$$\frac{dx}{dz}=3z^2,\ \frac{dz}{dx}=\frac{z^{-2}}{3}.$$

Hence
$$\frac{dy}{dx}=\frac{1}{3}z^{-2}\frac{dy}{dz}. \quad \dots\dots\dots\dots \quad (1)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx}.$$

From (1), $\frac{d}{dz}\left(\frac{dy}{dx}\right)=\frac{1}{3}\left(z^{-2}\frac{d^2y}{dz^2}-2z^{-3}\frac{dy}{dz}\right).$

Hence
$$\frac{d^2y}{dx^2}=\frac{1}{9}\left(z^{-4}\frac{d^2y}{dz^2}-2z^{-5}\frac{dy}{dz}\right). \quad \dots\dots\dots \quad (2)$$

Similarly, $$\frac{d^3y}{dx^3}=\frac{d}{dz}\left(\frac{d^2y}{dx^2}\right)\frac{dz}{dx}.$$

From (2), $$\frac{d}{dz}\left(\frac{d^2y}{dx^2}\right)=\frac{1}{9}\left(z^{-4}\frac{d^3y}{dz^3}-6\,z^{-5}\frac{d^2y}{dz^2}+10\,z^{-6}\frac{dy}{dz}\right).$$

Hence $$\frac{d^3y}{dx^3}=\frac{1}{27}\left(z^{-6}\frac{d^3y}{dz^3}-6\,z^{-7}\frac{d^2y}{dz^2}+10\,z^{-8}\frac{dy}{dz}\right).$$

EXAMPLES

Change the independent variable from x to y in the two following equations:

1. $3\left(\frac{d^2y}{dx^2}\right)^2-\frac{dy}{dx}\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2=0.$ *Ans.* $\frac{d^3x}{dy^3}+\frac{d^2x}{dy^2}=0.$

2. $\left(3a\frac{dy}{dx}+2\right)\left(\frac{d^2y}{dx^2}\right)^2=\left(a\frac{dy}{dx}+1\right)\frac{dy}{dx}\frac{d^3y}{dx^3}.$

Ans. $\left(\frac{d^2x}{dy^2}\right)^2=\left(\frac{dx}{dy}+a\right)\frac{d^3x}{dy^3}.$

Change the variable from y to z in the two following equations:

3. $\frac{d^2y}{dx^2}=1+\frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2,$ $y=\tan z.$

Ans. $\frac{d^2z}{dx^2}-2\left(\frac{dz}{dx}\right)^2=\cos^2 z.$

4. $(1+y)^2\left(\frac{d^3y}{dx^3}-2\,y\right)+\left(\frac{dy}{dx}\right)^3=2\,(1+y)\,\frac{dy}{dx}\frac{d^2y}{dx^2},$ $y=z^2+2\,z.$

Ans. $(z+1)\frac{d^3z}{dx^3}=\frac{dz}{dx}\frac{d^2z}{dx^2}+z^2+2\,z.$

Change the independent variable from x to z in the following equations:

5. $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0,$ $x^2=4\,z.$ *Ans.* $z\frac{d^2y}{dz^2}+\frac{dy}{dz}+y=0.$

6. $\frac{d^2y}{dx^2}+\frac{2\,x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0,$ $x=\tan z.$ *Ans.* $\frac{d^2y}{dz^2}+y=0$

7. $(2x-1)^3\dfrac{d^3y}{dx^3}+(2x-1)\dfrac{dy}{dx}=2y, \quad 2x=1+e^z.$

Ans. $4\dfrac{d^3y}{dz^3}-12\dfrac{d^2y}{dz^2}+9\dfrac{dy}{dz}-y=0.$

8. $x^4\dfrac{d^4y}{dx^4}+6x^3\dfrac{d^3y}{dx^3}+9x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=\log x, \quad x=e^z.$

Ans. $\dfrac{d^4y}{dz^4}+2\dfrac{d^2y}{dz^2}+y=z.$

120. Transformation of Partial Derivatives from Rectangular to Polar Coördinates.

Given $u=f(x,y),$

to express $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in terms of $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$, where x, y, are rectangular, and r, θ, polar coördinates.

We have from (5), Art. 113, regarding u as a function of r and θ,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}, \quad \ldots \ldots \ldots \quad (1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}. \quad \ldots \ldots \ldots \quad (2)$$

The values of $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial \theta}{\partial x}, \dfrac{\partial \theta}{\partial y}$, are now to be found from the relations between x, y, and r, θ.

These are $$x=r\cos\theta, \quad y=r\sin\theta. \quad \ldots \ldots \quad (3)$$

But in the partial derivatives $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}$, and $\dfrac{\partial \theta}{\partial x}, \dfrac{\partial \theta}{\partial y}$, r and θ are regarded as functions of x and y.

These are, from (3),

$$r^2=x^2+y^2, \quad \theta=\tan^{-1}\frac{y}{x}.$$

Differentiating, we find

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta,\quad \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta,$$

$$\frac{\partial\theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r},\quad \frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}.$$

Substituting in (1) and (2), we have

$$\frac{\partial u}{\partial x}=\cos\theta\,\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial u}{\partial\theta},\quad\ldots\ldots\ldots\quad(4)$$

$$\frac{\partial u}{\partial y}=\sin\theta\,\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial u}{\partial\theta}.\quad\ldots\ldots\ldots\quad(5)$$

121. Transformation of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ from Rectangular to Polar Coordinates. By substituting in (4), Art. 120, $\frac{\partial u}{\partial x}$ for u, we have

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\cos\theta\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)-\frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right).\quad\ldots\ldots\quad(1)$$

Differentiating (4), Art. 120, with respect to r,

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)=\cos\theta\frac{\partial^2 u}{\partial r^2}-\frac{\sin\theta}{r}\,\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\sin\theta}{r^2}\,\frac{\partial u}{\partial\theta}.\quad\ldots\ldots\quad(2)$$

Differentiating (4), Art. 120, with respect to θ,

$$\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)=\cos\theta\,\frac{\partial^2 u}{\partial r\partial\theta}-\sin\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial^2 u}{\partial\theta^2}-\frac{\cos\theta}{r}\,\frac{\partial u}{\partial\theta}.\quad\ldots\quad(3)$$

Substituting (2) and (3) in (1), we have

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Similarly by using (5), Art. 120, instead of (4), we find

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Adding (4) and (5), we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

CHAPTER XIII

MAXIMA AND MINIMA OF FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES

122. Definition. A function of two independent variables, $f(x, y)$, is said to have a maximum value when $x = a$, $y = b$; when, for all sufficiently small numerical values of h and k,

$$f(a, b) > f(a+h, b+k), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and a minimum value, when

$$f(a, b) < f(a+h, b+k). \quad . \qquad . \quad . \quad . \quad (b)$$

123. Conditions for Maxima or Minima.

If
$$u = f(x, y),$$

we find that a necessary condition for both (a) and (b) is that

$$\frac{\partial u}{\partial x} = 0, \text{ and } \frac{\partial u}{\partial y} = 0, \text{ when } x = a, \quad y = b.$$

This may be shown as follows:

Conditions (a) and (b) must hold when $k = 0$, and we have for a maximum

$$f(a, b) > f(a+h, b),$$

and for a minimum

$$f(a, b) < f(a+h, b),$$

for sufficiently small values of h.

We thus have for consideration a function of only one variable. By Art. 106, we must have for both maximum and minimum,

$$\frac{d}{dx}f(x, b) = 0, \text{ when } x = a,$$

that is

$$\frac{\partial}{\partial x}f(x, y) = 0, \text{ when } x = a,\ y = b.$$

Similarly, by letting $h = 0$ in (*a*) and (*b*), we may derive

$$\frac{\partial}{\partial y}f(x, y) = 0, \text{ when } x = a,\ y = b.$$

These conditions for a maximum or minimum are necessary but not sufficient. As in the case of maxima and minima of functions of one variable, there are additional conditions involving derivatives of higher orders. These we shall give without proof, as their rigorous derivation is beyond the scope of this book.

The conditions for a maximum or minimum value of $u = f(x, y)$ are as follows:

For either a maximum or minimum,

$$\frac{\partial u}{\partial x} = 0, \quad \text{and} \quad \frac{\partial u}{\partial y} = 0; \quad \ldots\ldots \quad (1)$$

also

$$\left(\frac{\partial^2 u}{\partial y\, \partial x}\right)^2 < \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}. \quad \ldots\ldots \quad (2)$$

For a maximum,

$$\frac{\partial^2 u}{\partial x^2} < 0, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} < 0. \quad \ldots\ldots \quad (3)$$

For a minimum,

$$\frac{\partial^2 u}{\partial x^2} > 0, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} > 0. \quad \ldots\ldots \quad (4)$$

124. Functions of Three Independent Variables. The conditions for a maximum or minimum value of $u = f(x, y, z)$ are as follows:

For either a maximum or minimum,

$$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial u}{\partial z} = 0,$$

and

$$\left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 < \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}.$$

For a maximum $\qquad \dfrac{\partial^2 u}{\partial x^2} < 0$, and $\Delta < 0$;

for a minimum, $\qquad \dfrac{\partial^2 u}{\partial x^2} > 0$, and $\Delta > 0$;

$$\text{where } \Delta = \begin{vmatrix} \dfrac{\partial^2 u}{\partial x^2}, & \dfrac{\partial^2 u}{\partial x\,\partial y}, & \dfrac{\partial^2 u}{\partial x\,\partial z} \\ \dfrac{\partial^2 u}{\partial x\,\partial y}, & \dfrac{\partial^2 u}{\partial y^2}, & \dfrac{\partial^2 u}{\partial y\,\partial z} \\ \dfrac{\partial^2 u}{\partial x\,\partial z}, & \dfrac{\partial^2 u}{\partial y\,\partial z}, & \dfrac{\partial^2 u}{\partial z^2} \end{vmatrix}.$$

EXAMPLES

1. Find the maximum value of

$$u = 3\,axy - x^3 - y^3.$$

Here $\qquad \dfrac{\partial u}{\partial x} = 3\,ay - 3\,x^2, \quad \dfrac{\partial u}{\partial y} = 3\,ax - 3\,y^2.$

Also $\qquad \dfrac{\partial^2 u}{\partial x^2} = -6\,x, \quad \dfrac{\partial^2 u}{\partial y^2} = -6\,y, \quad \dfrac{\partial^2 u}{\partial x\,\partial y} = 3\,a.$

Applying (1), Art. 123, we have

$$ay - x^2 = 0, \quad \text{and} \quad ax - y^2 = 0;$$

whence $\qquad x = 0,\ y = 0;$ or $x = a,\ y = a.$

The values $x = 0$, $y = 0$, give

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x\,\partial y} = 3\,a,$$

which do not satisfy (2), Art. 123.

Hence they do not give a maximum or minimum.

The values $x=a$, $y=a$, give

$$\frac{\partial^2 u}{\partial x^2}=-6a,\quad \frac{\partial^2 u}{\partial y^2}=-6a,\quad \frac{\partial^2 u}{\partial x\,\partial y}=3a,$$

which satisfy both (2) and (3), Art. 123.

Hence they give a maximum value of u, which is a^3.

2. Find the maximum value of xyz, subject to the condition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad . \quad . \quad . \quad . \qquad (1)$$

From (1) $$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2};$$

and as xyz is numerically a maximum when $x^2y^2z^2$ is a maximum we put

$$u=x^2y^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right).$$

$$\frac{\partial u}{\partial x}=2xy^2\left(1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}\right),\quad \frac{\partial u}{\partial y}=2x^2y\left(1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}\right),$$

$$\frac{\partial^2 u}{\partial x^2}=2y^2\left(1-\frac{6x^2}{a^2}-\frac{y^2}{b^2}\right),\quad \frac{\partial^2 u}{\partial y^2}=2x^2\left(1-\frac{x^2}{a^2}-\frac{6y^2}{b^2}\right),$$

$$\frac{\partial^2 u}{\partial x\,\partial y}=4xy\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}\right).$$

From $\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{dy}=0$, we find, as the only values satisfying (2), Art. 123,

$$x=\frac{a}{\sqrt{3}},\quad y=\frac{b}{\sqrt{3}},\quad \text{which give}$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{8b^2}{9},\quad \frac{\partial^2 u}{\partial y^2}=-\frac{8a^2}{9},\quad \frac{\partial^2 u}{\partial x\partial y}=-\frac{4ab}{9}.$$

As these values satisfy (2) and (3), Art. 123, it follows that xyz is a maximum when

$$x=\frac{a}{\sqrt{3}}, \quad y=\frac{b}{\sqrt{3}}, \quad z=\frac{c}{\sqrt{3}}.$$

The maximum value of xyz is $\frac{abc}{3\sqrt{3}}$.

3. Find the values of x, y, z that render

$$x^2+y^2+z^2+x-2z-xy$$

a minimum. *Ans.* $x=-\frac{2}{3}, \quad y=-\frac{1}{3}, \quad z=1.$

4. Find the maximum value of

$$(a-x)(a-y)(x+y-a).$$

Ans. $\frac{a^3}{27}$.

5. Find the minimum value of

$$x^2+xy+y^2-ax-by.$$

Ans. $\frac{1}{3}(ab-a^2-b^2)$.

6. Find the values of x and y that render

$$\sin x+\sin y+\cos(x+y)$$

a maximum or minimum. *Ans.* A minimum, when $x=y=\frac{3\pi}{2}$;

a maximum, when $x=y=\frac{\pi}{6}$ or $\frac{5\pi}{6}$.

7. Find the maximum value of

$$\frac{(ax+by+c)^2}{x^2+y^2+1}.$$

Ans. $a^2+b^2+c^2$.

8. Find the maximum value of $x^2y^3z^4$, subject to the condition

$$2x+3y+4z=a$$

Ans. $\left(\frac{a}{9}\right)^9$.

9. Find the minimum value of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$, subject to the condition

$$xyz=abc.$$

Ans. 3.

10. Divide a into three parts such that their continued product may be the greatest possible.

Let the parts be x, y, and $a-x-y$.

Then $$u = xy(a-x-y), \text{ to be a maximum.}$$

$$\frac{\partial u}{\partial x} = ay - 2xy - y^2 = 0, \quad \frac{\partial u}{\partial y} = ax - x^2 - 2xy = 0.$$

These equations give $x = y = \frac{a}{3}$.

Hence a is divided into equal parts.

NOTE. — When, from the nature of the problem, it is evident that there is a maximum or minimum, it is often unnecessary to consider the second derivatives.

11. Divide a into three parts, x, y, z, such that $x^m y^n z^p$ may be a maximum.

Ans. $\frac{x}{m} = \frac{y}{n} = \frac{z}{p} = \frac{a}{m+n+p}$.

12. Divide 30 into four parts such that the continued product of the first, the square of the second, the cube of the third, and the fourth power of the fourth, may be a maximum.

Ans. 3, 6, 9, 12.

13. Given the volume a^3 of a rectangular parallelopiped; find when the surface is a minimum.

Ans. When the parallelopiped is a cube.

14. An open vessel is to be constructed in the form of a rectangular parallelopiped, capable of containing 108 cubic inches of water. What must be its dimensions to require the least material in construction?

Ans. Length and width, 6 in.; height, 3 in.

15. Find the coördinates of a point, the sum of the squares of whose distances from three given points,

$$(x_1, y_1), \ (x_2\, y_2), \ (x_3, y_3),$$

is a minimum. *Ans.* $\frac{1}{3}(x_1+x_2+x_3),\ \frac{1}{3}(y_1+y_2+y_3),$

the centre of gravity of the triangle joining the given points.

16. If x, y, z are the perpendiculars from any point P on the sides a, b, c of a triangle of area Δ, find the minimum value of $x^2+y^2+z^2$.

Ans. $\frac{4\Delta^2}{a^2+b^2+c^2}.$

17. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad \textit{Ans.}\ \frac{8abc}{3\sqrt{3}}.$$

18. The electric time constant of a cylindrical coil of wire is

$$u=\frac{mxyz}{ax+by+cz},$$

where x is the mean radius, y is the difference between the internal and external radii, z is the axial length, and m, a, b, c are known constants. The volume of the coil is $nxyz=g$. Find the values of x, y, z which make u a minimum if the volume of the coil is fixed; also the minimum value of u.

Ans. $ax=by=cz=\sqrt[3]{\frac{abcg}{n}};\ u=\frac{m}{3}\sqrt[3]{\frac{g^2}{abcn^2}}.$

CHAPTER XIV

CURVES FOR REFERENCE

We give in this chapter representations and descriptions of some of the curves used as examples in the following chapters.

RECTANGULAR COÖRDINATES

125. The Cissoid,

$$y^2 = \frac{x^3}{2a - x}.$$

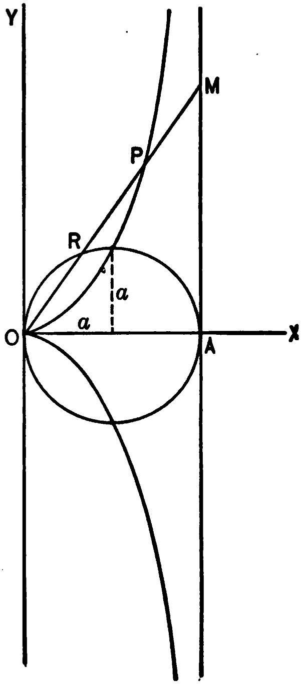
This curve may be constructed from the circle *ORA* (radius a) by drawing any oblique line *OM*, and making

$$PM = OR.$$

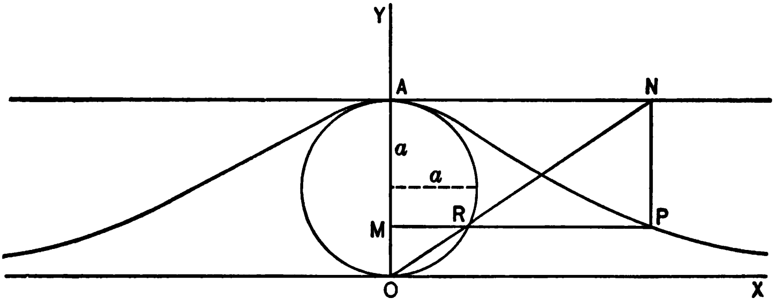
The equation above may be easily obtained from this construction. The line *AM* parallel to *OY* is an asymptote.

The polar equation of the cissoid is

$$r = 2a \sin\theta \tan\theta.$$



126. The Witch of Agnesi, $y = \dfrac{8a^3}{x^2 + 4a^2}$.



This curve may be constructed from the circle *ORA* (radius, a) by drawing any abscissa *MR*, and extending it to *P* determined by *ORN*, by the construction shown in the figure.

The equation above may be derived from this construction. The axis of *X* is an asymptote.

127. The Folium of Descartes,

$$x^3 + y^3 - 3axy = 0.$$

The point *A*, the vertex of the loop, is

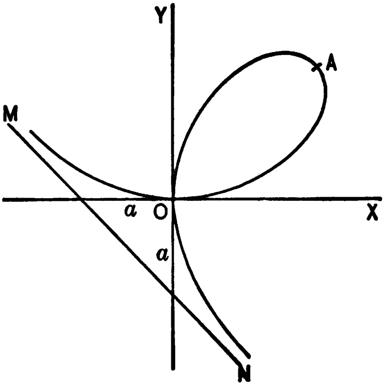
$$\left(\frac{3a}{2}, \frac{3a}{2}\right).$$

The equation of the asymptote *MN* is

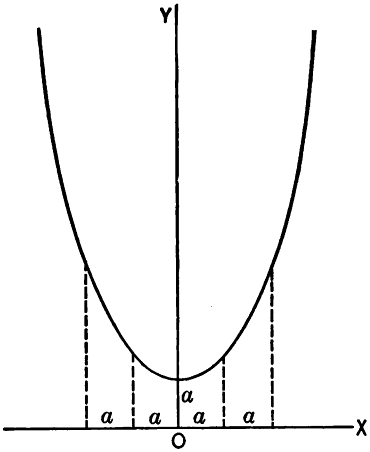
$$x + y + a = 0.$$

The polar equation of the folium is

$$r = \frac{3a \tan\theta \sec\theta}{1 + \tan^3\theta}.$$



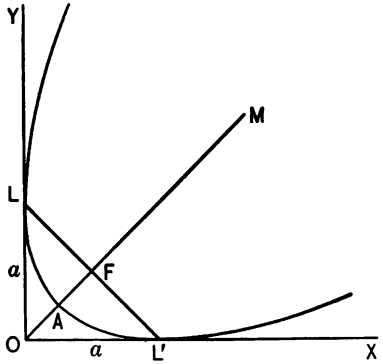
128. **The Catenary,** $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$.



This is the curve of a cord or chain suspended freely between two points.

129. **The Parabola, referred to Tangents at the Extremities of the Latus Rectum,** $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$.

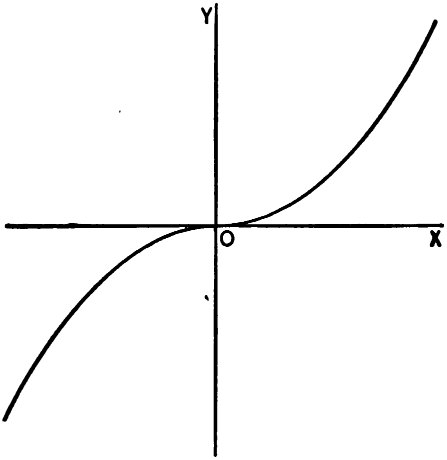
$$OL=OL'=a.$$



The line *LL'* is the latus rectum; its middle point *F*, the focus; *OFM*, the axis of the parabola; *A* the middle point of *OF*, the vertex.

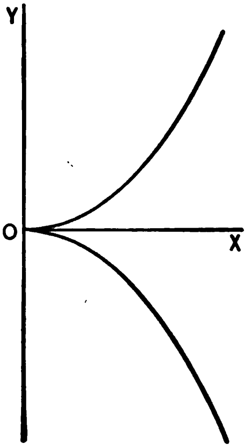
130. The curve $a^{n-1}y = x^n$, where one coördinate is proportional to the nth power of the other, is sometimes called the *parabola of the nth degree.*

If $n = 3$, we have the *Cubical Parabola,* $a^2y = x^3$.



If $n = \frac{3}{2}$, we have the *Semicubical Parabola,*

$$a^{\frac{1}{2}}y = x^{\frac{3}{2}},\ ay^2 = x^3.$$

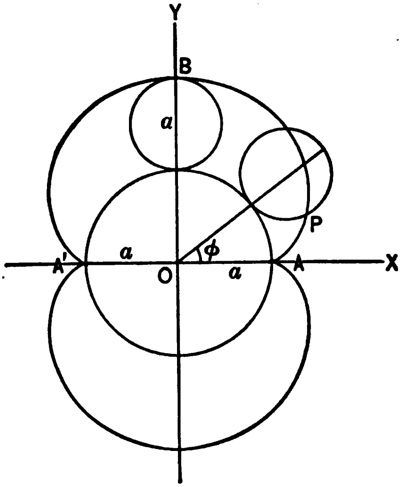


131. The Two-arched Epicycloid.

$$x = \frac{3a}{2}\cos\phi - \frac{a}{2}\cos 3\phi,$$

$$y = \frac{3a}{2}\sin\phi - \frac{a}{2}\sin 3\phi.$$

See (3), Art. 293.



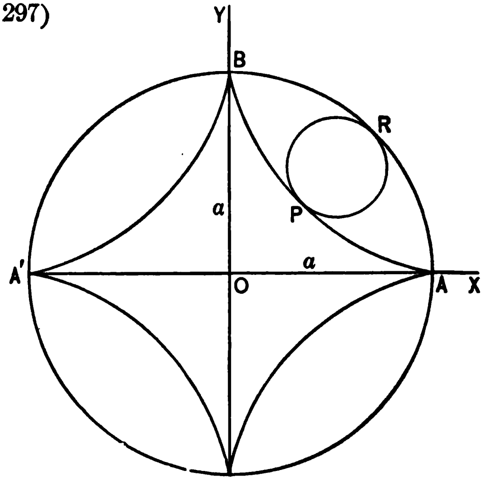
132. The Hypocycloid of Four Cusps sometimes called the Astroid,

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (Art. 297)

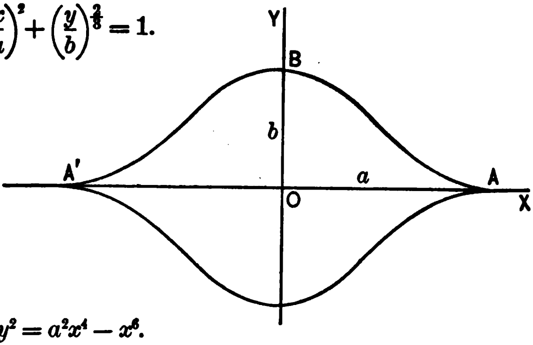
This is the curve described by a point *P* in the circumference of the circle *PR*, as it rolls within the circumference of the fixed circle *ABA'*, whose radius *a* is four times that of the former.

The equation above may be given in the form

$x = a\cos^3\phi$, $y = a\sin^3\phi$.

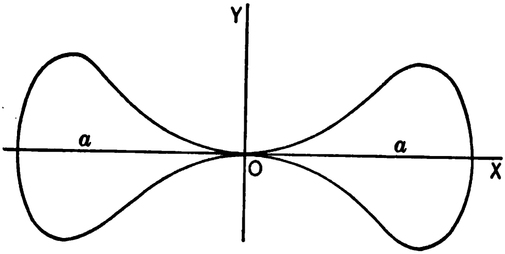


133. The Curve, $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$



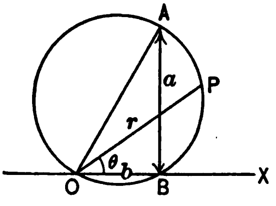
The equation is the same as that of the ellipse with the exponent of the second term changed from 2 to $\frac{2}{3}$.

134. The Curve, $a^4y^2 = a^2x^4 - x^6.$



POLAR COÖRDINATES

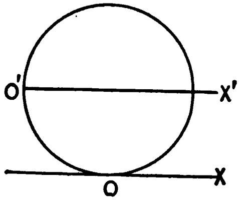
135. The Circle, $r = a \sin \theta + b \cos \theta.$



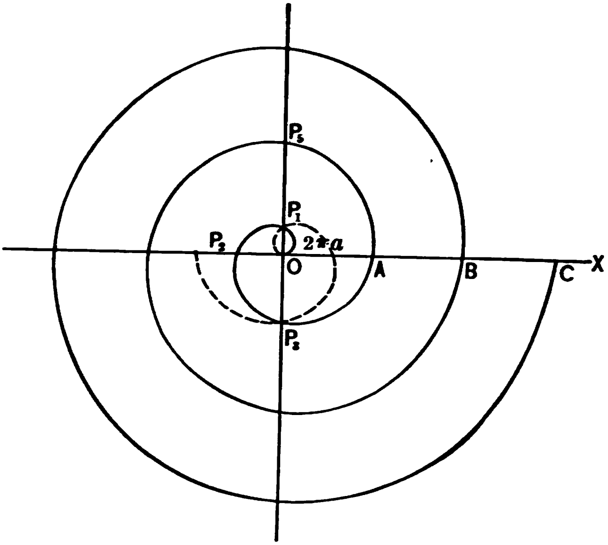
By laying off $OB = b$, and $BA = a$, we determine OA the diameter $= \sqrt{a^2 + b^2}$.

If $b = 0$, $r = a \sin \theta$, the circle referred to OX. (2d fig.)

If $a = 0$, $r = b \cos \theta$, the circle referred to $O'X'$.



136. The Spiral of Archimedes, $r = a\theta$.



In this curve r is proportional to θ. Laying off

$$r = OA,$$

when

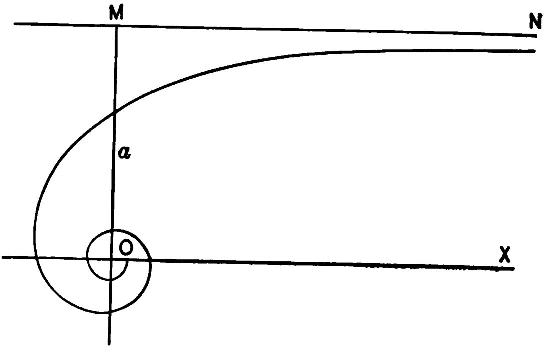
$$\theta = 2\pi,$$

then

$$OP_1 = \tfrac{1}{4}\,OA,\quad OP_2 = \tfrac{1}{2}\,OA,\quad OP_3 = \tfrac{3}{4}\,OA,\quad OP_5 = \tfrac{5}{4}\,OA,\ \cdots,$$
$$OB = 2\,OA,\quad OC = 3\,OA.$$

The dotted portion corresponds to negative values of θ.

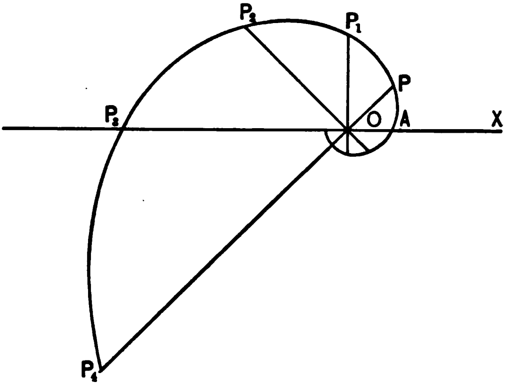
137. The Hyperbolic or Reciprocal Spiral, $r\theta = a$.



In this curve r varies inversely as θ. The line MN is an asymptote, which the curve approaches, as θ approaches zero.

Since $r=0$ only when $\theta=\infty$, it follows that an infinite number of revolutions are necessary to reach the origin.

138. The Logarithmic Spiral, $r = e^{a\theta}$.

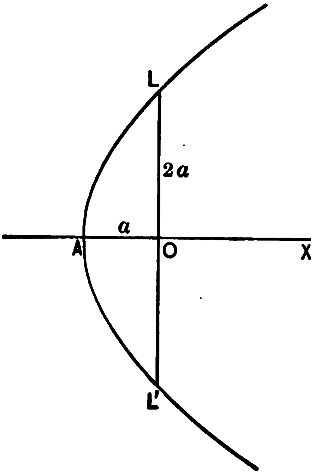


Starting from A, where $\theta=0$ and $r=1$, r increases with θ; but if we suppose θ negative, r decreases as θ numerically increases. Since $r=0$ only when $\theta=-\infty$, it follows that an infinite number of retrograde revolutions from A is required to reach the origin O.

A property of this spiral is that the radii vectores OP, OP_1, OP_2, $\cdots$, make a constant angle with the curve.

139. The Parabola, Origin at Focus, $r(1-\cos\theta) = 2a$.

That is, $r = a \operatorname{cosec}^2 \frac{\theta}{2}$.



The initial line OX is the axis of the parabola; the origin O is the focus; LL', the latus rectum.

If the parabola is revolved 180° about O, its equation is then

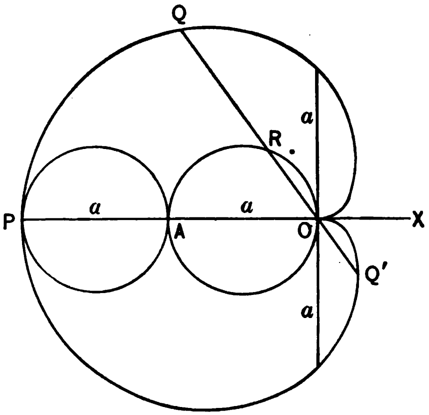
$$r = a \sec^2 \frac{\theta}{2}.$$

140. The Parabola, Origin at Vertex (see preceding figure),

$$r \sin\theta \tan\theta = 4a.$$

The initial line is the axis AX; the origin is the vertex A.

141. The Cardioid, $r = a(1 - \cos\theta)$. (Art. 295)



This is the curve described by a point P in the circumference of a circle PA (diameter, a) as it rolls upon an equal fixed circle OA.

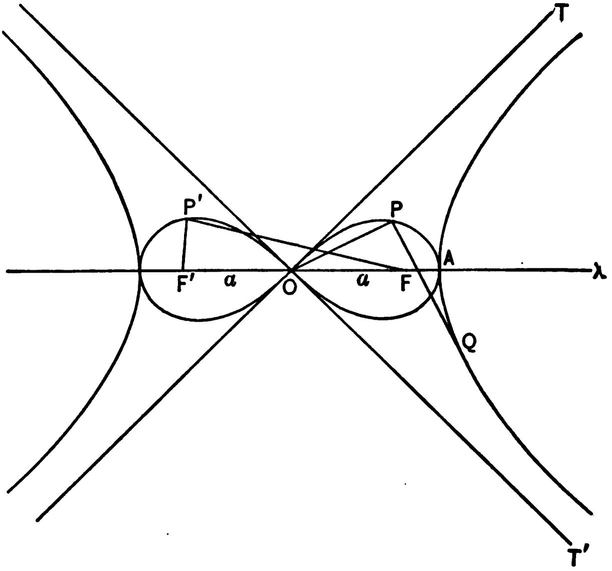
Or it may be constructed by drawing through O, any line OR in the circle OA, and producing OR to Q and Q', making $RQ = RQ' = OA$.

The given equation follows directly from this construction.

If the cardioid is revolved 90° to the right about O, its equation is

$$r = a(1 + \sin\theta).$$

142. The Equilateral Hyperbola, $r^2 \cos 2\theta = a^2$.



The origin O is the centre of the hyperbola, and the initial line OX is the transverse axis.

If OT' is taken as the initial line, the equation of the hyperbola is

$r^2 \sin 2\theta = a^2$.

143. The Lemniscate referred to OA (see preceding figure),

$$r^2 = a^2 \cos 2\theta.$$

This is a curve of two loops like the figure eight.

It may be defined in connection with the equilateral hyperbola, as the locus of P, the foot of a perpendicular from O on PQ, any tangent to the hyperbola.

The loops are limited by the asymptotes of the hyperbola, making

$$TOX = T'OX = 45^\circ. \quad OA = a.$$

The lemniscate has the following property:

If two points, F and F', called the foci, be taken on the axis, such that

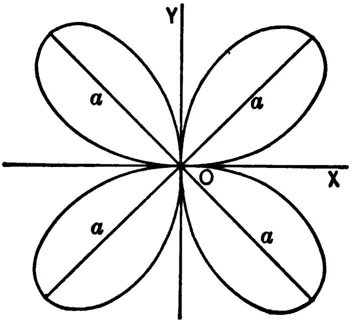
$$OF = OF' = \frac{a}{\sqrt{2}},$$

then the product of the distances $P'F$, $P'F'$, of any point of the curve from these fixed points, is constant, and equal to the square of OF.

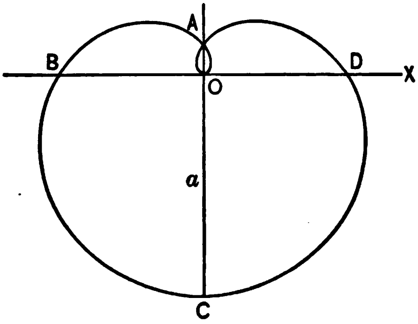
If OT' is taken as the initial line, the equation of the lemniscate is

$$r^2 = a^2 \sin 2\theta.$$

144. The Four-leaved Rose, $r = a \sin 2\theta$.



145. The Curve, $r = a \sin^3 \frac{\theta}{3}$.

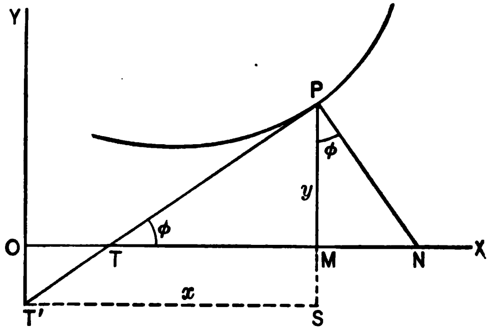


CHAPTER XV

DIRECTION OF CURVES. TANGENTS AND NORMALS

We have seen in Art. 17 that the derivative at any point of a plane curve is the slope of the curve at that point. We will now consider some further applications of differentiation to curves.

146. Subtangent, Subnormal, Intercepts of Tangent. — Let PT be the tangent, and PN the normal, to a curve at the point P, whose ordinate is $y = PM$. Then MT is called the *subtangent*, and MN the *subnormal*, corresponding to the point P.



To find expressions for these quantities:

Let ϕ donote the angle PTX, the inclination of the tangent to OX.

By Art. 17,
$$\tan PTX = \tan \phi = \frac{dy}{dx}.$$

$$\text{Subtangent} = TM = PM \cot PTM = y \cot \phi = \frac{y}{\frac{dy}{dx}} = y\frac{dx}{dy}.$$

$$\text{Subnormal} = MN = PM \tan MPN = y \tan \phi = y\frac{dy}{dx}.$$

$$\text{Intercept of tangent on } OX = OT = OM - TM = x - y\frac{dx}{dy}.$$

$$\text{Intercept of tangent on } OY = OT' = PS - PM = x \tan \phi - y.$$

But as OT' is negative, we have

$$\text{Intercept of tangent on } OY = y - x \tan \phi = y - x\frac{dy}{dx}.$$

147. Angle of Intersection of Two Curves. Suppose the two curves intersect at P.

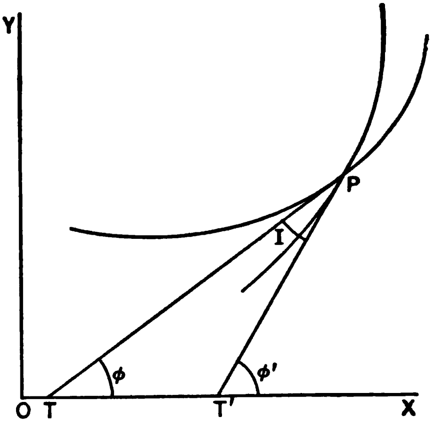
Let PT and PT' be the tangents at P.

$$PTX=\phi,\ PT'X=\phi',$$

and let I be the angle TPT' between the tangents.

Then $I=\phi'-\phi$ and

$$\tan I=\frac{\tan\phi'-\tan\phi}{1+\tan\phi'\tan\phi}. \quad (1)$$



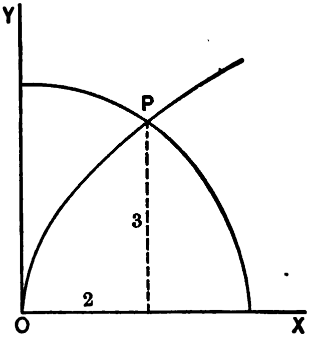
From the equations of the given curves find the coördinates of the point of intersection P; then using these equations separately, find by $\tan\phi=\frac{dy}{dx}$ the values of $\tan\phi$ and $\tan\phi'$ for the point P. Substituting in (1) gives $\tan I$.

For example find the angle at which the circle

$$x^2+y^2=13, \quad \ldots\ldots\ldots \quad (2)$$

intersects the parabola

$$2y^2=9x. \quad \ldots\ldots\ldots \quad (3)$$



The intersection P of (2) and (3) is found to be (2, 3).

Differentiating (2),

$$\frac{dy}{dx}=-\frac{x}{y}=-\frac{2}{3} \text{ for } P,\ \tan\phi=-\frac{2}{3}.$$

From (3), $$\frac{dy}{dx}=\frac{9}{4y}=\frac{3}{4} \text{ for } P, \qquad \tan\phi'=\frac{3}{4}.$$

Substituting in (1), $\tan I=\frac{17}{6},\ I=70°\,33'.$

EXAMPLES

1. Find the direction at the origin of the curve,

$$(a^4 - b^4)\,y = x\,(x - a)^4 - b^4 x. \qquad \textit{Ans.}\ 45^\circ \text{ with } OX.$$

What must be the relation between a and b, so that it may be parallel to OX at the point $x = 2a$? *Ans.* $3a^2 = b^2$.

2. Find the points of contact of the two tangents to the curve,

$$6y = 2x^3 + 9x^2 - 12x + 2,$$

parallel to the tangent at the origin to the curve, $y^2 + ay = 2ax$. *Ans.* $\left(1, \frac{1}{6}\right)$, $(-4, 11)$.

3. Find the subtangents and subnormals in the parabolas,

$$y^2 = 4ax, \text{ and } x^2 = 4ay.$$

Ans. Subtangents, $2x, \frac{x}{2}$; subnormals, $2a, \frac{x^3}{8a^2}$.

4. Find the subtangent and subnormal in the cissoid (Art. 125),

$$y^2 = \frac{x^3}{2a - x}, \text{ at the point } (a, a). \qquad \textit{Ans.}\ \frac{a}{2},\ 2a.$$

5. Show that the sum of the intercepts of the tangent to the parabola (Art. 129), $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, is equal to a.

6. Show that the area of the triangle intercepted from the coördinate axes by the tangent to the hyperbola,

$$2xy = a^2, \text{ is equal to } a^2.$$

7. Show that the part of the tangent to the hypocycloid (Art. 132), $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, intercepted between the coördinate axes, is equal to a.

8. At what angle do the parabolas, $y^2 = ax$ and $x^2 = 8ay$ intersect?

Ans. At $(0, 0)$, 90°; at another point, $\tan^{-1}\frac{3}{5}$.

9. At what angle does the circle, $x^2+y^2=5x$, intersect the curve, $3y=7x^3-1$, at their common point (1, 2)? *Ans.* 45°.

10. Show that the ellipse and hyperbola,

$$\frac{x^2}{7}+\frac{y^2}{2}=1, \quad \frac{x^2}{3}-\frac{y^2}{2}=1,$$

intersect at right angles.

11. Find the angle of intersection of the circles,

$x^2+y^2-x+3y+2=0, \quad x^2+y^2-2y=9.$ *Ans.* $\tan^{-1}\frac{1}{2}$.

12. Show that the parabola and ellipse,

$$y^2=ax, \quad 2x^2+y^2=b^2,$$

intersect at right angles.

13. Show that the parabolas,

$$y^2=2ax+a^2, \text{ and } x^2=2by+b^2,$$

intersect at an angle of 45°.

14. Find the angle of intersection of the parabola,

$x^2=4ay$, and the witch (Art. 126), $y=\dfrac{8a^3}{x^2+4a^2}$.

Ans. $\tan^{-1}3=71°\ 34'$.

15. Find the angle of intersection between the parabola, $y^2=4ax$, and its evolute, $27ay^2=4(x-2a)^3$. (See Fig., Art. 167.)

Ans. $\tan^{-1}\dfrac{1}{\sqrt{2}}$.

148. Equations of the Tangent and Normal. Having given the equation of a curve $y=f(x)$, let it be required to find the equation of a straight line tangent to it at a given point.

Let (x', y') be the given point of contact. Then the equation of a straight line through this point is

$$y - y' = m(x - x'), \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

in which x and y are the variable coördinates of any point in the straight line; and m, the tangent of its inclination to the axis of X. But since the line is to be tangent to the given curve, we must have, by Art. 17,

$$m = \tan\phi = \frac{dy}{dx},$$

$\frac{dy}{dx}$ being derived from the equation of the given curve $y = f(x)$, and applied to the point of contact (x', y').

If we denote this by $\frac{dy'}{dx'}$, we have, substituting $m = \frac{dy'}{dx'}$ in equation (1),

$$y - y' = \frac{dy'}{dx'}(x - x'), \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

for the equation of the required tangent.

Since the normal is a line through (x', y') perpendicular to the tangent, we have for its equation

$$y - y' = -\frac{1}{\frac{dy'}{dx'}}(x - x') = -\frac{dx'}{dy'}(x - x'). \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

For example, find the equations of the tangent and normal to the circle $x^2 + y^2 = a^2$, at the point (x', y').

Here, by differentiating $x^2 + y^2 = a^2$, we find

$$\frac{dy}{dx} = -\frac{x}{y}, \text{ from which } \frac{dy'}{dx'} = -\frac{x'}{y'}.$$

Substituting in (2), we have

$$y - y' = -\frac{x'}{y'}(x - x'),$$

as the equation of the required tangent.

It may be simplified as follows:

$$yy' - y'^2 = -xx' + x'^2,$$

$$xx' + yy' = x'^2 + y'^2 = a^2.$$

The equation of the normal to the circle is found from (3) to be

$$y - y' = \frac{y'}{x'}(x - x'),$$

which reduces to $x'y = y'x.$

EXAMPLES

Find the equations of the tangent and normal to each of the three following curves at the point (x', y'):

1. The parabola, $y^2 = 4\,ax.$

Ans. $yy' = 2\,a(x + x'),\ 2\,a(y - y') + y'(x - x') = 0.$

2. The ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Ans. $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,\ b^2x'(y - y') = a^2y'(x - x').$

3. The equilateral hyperbola, $2\,xy = a^2.$

Ans. $xy' + yx' = a^2,\ y'(y - y') = x'(x - x').$

4. Find the equation of the tangent at the point (x', y') to the ellipse, $3\,x^2 - 4\,xy + 2\,y^2 + 2\,x = 2.$

Ans. $3\,xx' + 2\,yy' - 2\,(x'y + y'x) + x + x' = 2.$

5. Find the equations of tangent and normal at the point (x', y') to the curve, $x^5 = a^3y^2.$

Ans. $\frac{5\,x}{x'} - \frac{2\,y}{y'} = 3,\quad 2\,xx' + 5\,yy' = 2\,x'^2 + 5\,y'^2.$

6. In the cissoid (Art. 125), $y^2 = \frac{x^3}{2a - x}$, find the equations of the tangent and normal at the points whose abscissa is a.

Ans. At (a, a), $y = 2x - a$, $2y + x = 3a$.
At $(a, -a)$, $y + 2x = a$, $2y = x - 3a$.

7. In the witch (Art. 126), $y = \frac{8a^3}{4a^2 + x^2}$, find the equations of the tangent and normal at the point whose abscissa is $2a$.

Ans. $x + 2y = 4a$, $y = 2x - 3a$.

8. Find the equation of the tangent at the point (x', y') to the curve, $x^2y + xy^2 = a^3$.

Ans. $xy'(2x' + y') + yx'(2y' + x') = 3a^3$.

Find the equations of tangent and normal to the three following curves:

9. $x^3 + y^3 = 3axy$ (Art. 127), at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$. *Ans.* $x + y = 3a$, $x = y$.

10. $x + y = 2e^{x-y}$, at $(1, 1)$. *Ans.* $3y = x + 2$, $3x + y = 4$.

11. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$, at (a, b). *Ans.* $\frac{x}{a} + \frac{y}{b} = 2$, $ax - by = a^2 - b^2$.

12. Find the equations of the two tangents to the circle,
$x^2 + y^2 - 3y = 14$, parallel to the line, $7y = 4x + 1$.

Ans. $7y = 4x + 43$, $7y = 4x - 22$.

13. Find the equations of the two normals to the hyperbola,
$4x^2 - 9y^2 + 36 = 0$, parallel to the line, $2y + 5x = 0$.

Ans. $8y + 20x = \pm 65$.

149. Asymptotes.[*] When the tangent to a curve approaches a limiting position, as the distance of the point of contact from the origin is indefinitely increased, this limiting position is called an asymptote. In other words, an asymptote is a tangent which passes within a finite distance of the origin, although its point of contact is at an infinite distance.

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[*] The limits of this work allow only a brief notice of this subject.

We have found in Art. 146, for the intercepts of the tangent on the coördinate axes,

$$\text{Intercept on } OX = x - y\frac{dx}{dy}, \quad \text{Intercept on } OY = y - x\frac{dy}{dx}.$$

If either of these intercepts is finite for $x=\infty$, or $y=\infty$, the corresponding tangent will be an asymptote.

The equation of this asymptote may be obtained from its two intercepts, or from one intercept and the limiting value of $\frac{dy}{dx}$.

Let us investigate the conic sections with reference to asymptotes.

(1) The parabola, $y^2 = 4\,ax$, $\quad \frac{dy}{dx} = \frac{2\,a}{y}$.

$$\text{Intercept on } OX = x - y\frac{dx}{dy} = x - \frac{y^2}{2\,a} = -x,$$

$$\text{Intercept on } OY = y - x\frac{dy}{dx} = y - \frac{2\,ax}{y} = \frac{y}{2}.$$

When $x=\infty$, $y=\infty$, and both intercepts are also infinite. Hence the parabola has no asymptote.

(2) The hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\quad \frac{dy}{dx} = \frac{b^2x}{a^2y}$.

$$\text{Intercept on } OX = \frac{a^2}{x}, \quad \text{Intercept on } OY = -\frac{b^2}{y}.$$

These intercepts are both zero when $x=\infty$, and there is an asymptote passing through the origin. To find its equation, it is necessary to find the limiting value of $\frac{dy}{dx}$, when $x=\infty$.

$$\frac{dy}{dx} = \frac{b^2x}{a^2y} = \pm\frac{bx}{a\sqrt{x^2-a^2}} = \pm\frac{b}{a}\frac{1}{\sqrt{1-\frac{a^2}{x^2}}}.$$

Hence $$\frac{dy}{dx} = \pm\frac{b}{a}, \quad \text{when } x=\infty.$$

There are then two asymptotes, whose equations are

$$y = \pm \frac{b}{a}x.$$

(3) The ellipse, having no infinite branches, can have no asymptote.

150. Asymptotes Parallel to the Coördinate Axes. When, in the equation of the curve, $x = \infty$ gives a finite value of y, as $y = a$, then $y = a$ is the equation of an asymptote parallel to OX.

And when $y = \infty$ gives $x = a$, then $x = a$ is an asymptote parallel to OY.

151. Asymptotes by Expansion. Frequently an asymptote may be determined by solving the equation of the curve for x or y, and expanding the second member.

For example, to find the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$y = \pm \frac{b}{a}(x^2 - a^2)^{\frac{1}{2}} = \pm \frac{bx}{a}\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}} = \pm \frac{bx}{a}\left(1 - \frac{a^2}{2x^2} - \cdots\right).$$

As x increases indefinitely, the curve approaches the lines $y = \pm \frac{bx}{a}$, the asymptotes.

EXAMPLES

Investigate the following curves with reference to asymptotes:

1. $y = \frac{x^3}{x^2 + 3a^2}$. Asymptote, $y = x$.

2. $y^3 = 6x^2 - x^3$. Asymptote, $x + y = 2$.

3. The cissoid (Art. 125) $y^2 = \frac{x^3}{2a - x}$ Asymptote, $x = 2a$.

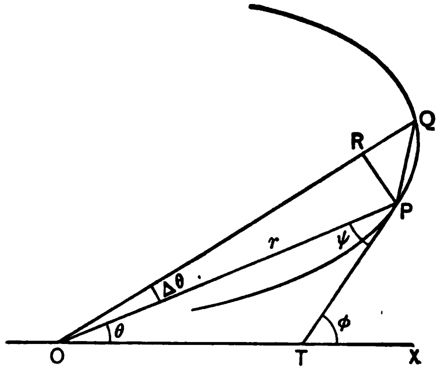
4. $x^3+y^3=a^3$. Asymptote, $x+y=0$.

5. $(x-2a)y^2=x^3-a^3$. Asymptotes, $x=2a$, $x+a=\pm y$.

6. $x^3+y^3=3axy$ (Art. 127). Asymptote, $x+y+a=0$.

(Substitute $y=vx$ in the given equation and in the expressions for the intercepts.)

152. Direction of Curve. Polar Coördinates.



In this case the angle OPT between the tangent and the radius vector may be most readily obtained. Denote this angle by ψ. Let r, θ, be the coördinates of P; $r+\Delta r, \theta+\Delta\theta$, the coördinates of Q. Draw PR perpendicular to OQ.

Then $$\tan PQR=\frac{PR}{RQ}=\frac{r\sin\Delta\theta}{r+\Delta r-r\cos\Delta\theta}=\frac{r\sin\Delta\theta}{\Delta r+2r\sin^2\dfrac{\Delta\theta}{2}}$$

$$=\frac{r\dfrac{\sin\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta}+r\sin\dfrac{\Delta\theta}{2}\dfrac{\sin\dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}}}.$$

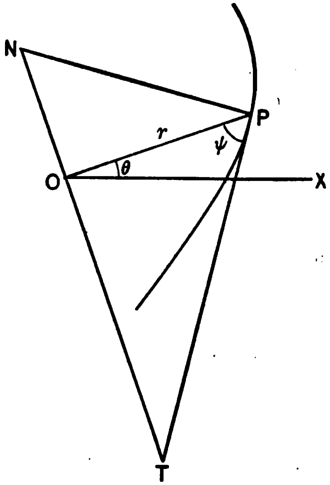
Now let $\Delta\theta$ approach zero; the point Q approaches P, and the angle PQR approaches its limit ψ.

Hence $$\tan\psi=\text{Lim}_{\Delta\theta=0}\tan PQR=\frac{r}{\dfrac{dr}{d\theta}}. \quad \cdot\quad\cdot\quad\cdot\quad\cdot\quad (1)$$

The inclination ϕ of the tangent to OX may be found by

$$\phi=\psi+\theta. \quad \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad (2)$$

153. Polar Subtangent and Subnormal.



If through O, NT be drawn perpendicular to OP, OT is called the *polar subtangent*, and ON the *polar subnormal*, corresponding to the point P.

$OT = OP \tan OPT$; that is,

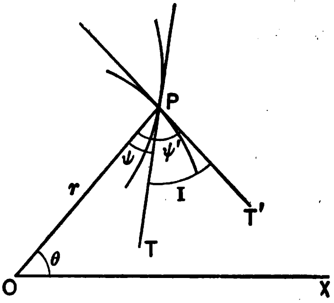
$$\text{Polar subtangent} = r\tan\psi = \frac{r^2}{\frac{dr}{d\theta}}.$$

$ON = OP \cot PNO$; that is,

$$\text{Polar subnormal} = r\cot\psi = \frac{dr}{d\theta}.$$

154. Angle of Intersection. Suppose the two curves intersect at P, and have the tangents PT and PT'.

$$OPT = \psi, \quad OPT' = \psi'.$$



Then the angle of intersection,

$$I = \psi' - \psi,$$

and $$\tan I = \frac{\tan\psi' - \tan\psi}{1 + \tan\psi' \tan\psi}. \qquad (1)$$

By this formula the angle of intersection may be found in polar coördinates, in the same way as by (1), Art. 147, in rectangular coördinates.

For example, find the angle of intersection between the curves

$$r = a\sin 2\theta, \qquad (2)$$

and $$r = a\cos 2\theta. \qquad (3)$$

From (2) and (3) we have for the intersection

$$\tan 2\theta = 1.$$

From (2), $\tan\psi' = \frac{1}{2}\tan 2\theta = \frac{1}{2}$, for the intersection.

From (3), $\tan\psi = -\frac{1}{2}\cot 2\theta = -\frac{1}{2}$, for the intersection.

Substituting in (1), $\tan I = \frac{4}{3}$.

The curves are that in Art. 144, and the same curve revolved 45° about the origin.

EXAMPLES

1. In the circle (Art. 135), $r = a\sin\theta$, find ψ and ϕ.

Ans. $\psi = \theta$, and $\phi = 2\theta$.

2. In the logarithmic spiral (Art. 138), $r = e^{a\theta}$, show that ψ is constant.

3. In the spiral of Archimedes (Art. 136), $r = a\theta$, show that $\tan\psi = \theta$; thence find the values of ψ, when $\theta = 2\pi$ and 4π.

Ans. 80° 57′ and 85° 27′.

Also show that the polar subnormal is constant.

4. The equation of the lemniscate (Art. 143) referred to a tangent at its center is $r^2 = a^2\sin 2\theta$. Find ψ, ϕ, and the polar subtangent.

Ans. $\psi = 2\theta$; $\phi = 3\theta$; subtangent $= a\tan 2\theta\sqrt{\sin 2\theta}$.

5. In the cardioid (Art. 141), $r = a(1 - \cos\theta)$, find ϕ, ψ, and the polar subtangent.

Ans. $\phi = \frac{3\theta}{2}$; $\psi = \frac{\theta}{2}$; subtangent $= 2a\tan\frac{\theta}{2}\sin^2\frac{\theta}{2}$.

6. Find the area of the circumscribed square of the preceding cardioid, formed by tangents inclined 45° to the axis.

Ans. $\frac{27}{16}(2+\sqrt{3})a^2$.

7. In the folium of Descartes (Art. 127), $r = \dfrac{3a\tan\theta\sec\theta}{1+\tan^3\theta}$,

show that
$$\tan\phi = \frac{\tan^4\theta - 2\tan\theta}{2\tan^3\theta - 1}.$$

8. Find the area of the square circumscribed about the loop of the folium of the preceding example.

Ans. $2\sqrt[3]{2}\,a^2$.

9. Show that the spiral of Archimedes (Art. 136), $r = a\theta$, and the reciprocal spiral (Art. 137), $r\theta = a$, intersect at right angles.

10. Show that the cardioids (Art. 141),

$$r = a(1 - \cos\theta), \quad r = b(1 + \sin\theta),$$

intersect at an angle of 45°.

11. Show that the parabolas (Art. 139),

$$r = m\sec^2\frac{\theta}{2}, \quad r = n\operatorname{cosec}^2\frac{\theta}{2},$$

intersect at right angles.

12. Find the angle of the intersection between the circle (Art. 135), $r = a\sin\theta$, and the curve (Art. 144), $r = a\sin 2\theta$.

Ans. At origin 0°; at two other points, $\tan^{-1}3\sqrt{3} = 79°\,6'$.

13. Find the angle of intersection between the circle (Art. 135), $r = 2a\cos\theta$, and the cissoid (Art. 125), $r = 2a\sin\theta\tan\theta$.

Ans. $\tan^{-1}2$.

14. At what angle does the straight line, $r\cos\theta = 2a$, intersect the circle (Art. 135), $r = 5a\sin\theta$?

Ans. $\tan^{-1}\frac{3}{4}$.

15. Show that the equilateral hyperbolas (Art. 142), $r^2\sin 2\theta = a^2$, $r^2\cos 2\theta = b^2$, intersect at right angles.

16. Find the angle of intersection between the circles (Art. 135)

$$r = a\sin\theta + b\cos\theta, \quad r = a\cos\theta + b\sin\theta.$$

Ans. $\tan^{-1}\dfrac{a^2 - b^2}{2ab}$.

17. Find the angle of intersection between the lemniscate (Art. 143), $r^2 = a^2\sin 2\theta$, and the equilateral hyperbola (Art. 142), $r^2 \sin 2\theta = b^2$.

Ans. $2\sin^{-1}\dfrac{b}{a}$.

155. Derivative of an Arc. Rectangular Coördinates. Let s denote the length of the arc of the curve measured from any fixed point of it.

Then $$s = \text{arc } AP, \quad \Delta s = \text{arc } PQ.$$

We have $$\sec QPR = \frac{PQ}{PR}.$$

Now suppose Δx to approach zero, and consequently the point Q to approach P.

Then

$$\text{Lim} \sec QPR = \sec TPR = \sec\phi.$$

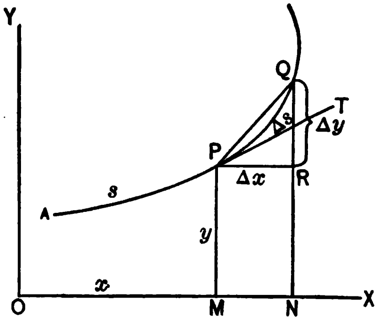
$$\frac{PQ}{PR} = \frac{PQ}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{PR}.$$

Since

$$\text{Lim} \frac{PQ}{\text{arc } PQ} = 1,$$

$$\text{Lim} \frac{PQ}{PR} = \text{Lim} \frac{\text{arc } PQ}{PR}$$

$$= \text{Lim} \frac{\Delta s}{\Delta x} = \frac{ds}{dx}.$$



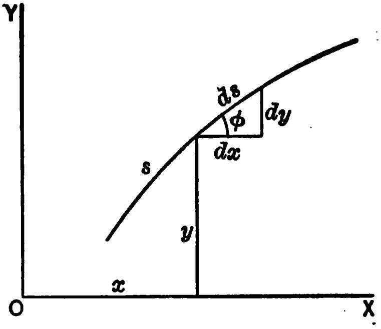
Hence $$\sec\phi = \frac{ds}{dx};$$

therefore $$\frac{ds}{dx} = \sqrt{1 + \tan^2\phi} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \quad \ldots\ldots \quad (1)$$

It is evident also that

$$\sin\phi=\frac{dy}{ds},\quad \cos\phi=\frac{dx}{ds}.\quad .\quad (2)$$

It may be noticed that these relations (1) and (2) are correctly represented by a right triangle, whose hypothenuse is *ds*, sides *dx* and *dy*, and angle at the base ϕ.



Here $$ds=\sqrt{(dx)^2+(dy)^2},$$

or $$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

156. Derivative of an Arc. Polar Coördinates. From the figure of Art. 152, we have, as $\Delta\theta$ approaches zero,

$$\sec\psi=\text{Lim}\sec PQR=\text{Lim}\,\frac{PQ}{RQ}=\text{Lim}\,\frac{\text{arc}\,PQ}{RQ}=\text{Lim}\,\frac{\Delta s}{RQ}.$$

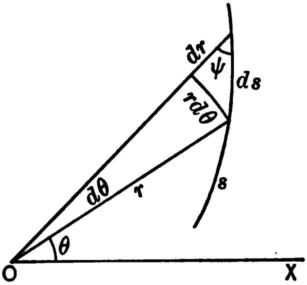
$$\frac{\Delta s}{RQ}=\frac{\Delta s}{\Delta r+2r\sin^2\frac{\Delta\theta}{2}}=\frac{\frac{\Delta s}{\Delta\theta}}{\frac{\Delta r}{\Delta\theta}+r\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\sin\frac{\Delta\theta}{2}}.$$

$$\sec\psi=\text{Lim}\,\frac{\Delta s}{RQ}=\frac{\frac{ds}{d\theta}}{\frac{dr}{d\theta}}=\frac{ds}{dr}.\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (1)$$

Hence $$\frac{ds}{dr}=\sqrt{1+\tan^2\psi}=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2},\quad .\ .\ .\ .\quad (2)$$

$$\frac{ds}{d\theta}=\frac{ds}{dr}\frac{dr}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (3)$$

It may be noticed that these relations (1), (2), and (3), are correctly represented by a right triangle, whose hypothenuse is ds, sides dr and $rd\theta$, and angle between dr and ds, ψ.



Here $$ds = \sqrt{(dr)^2 + (r\,d\theta)^2},$$

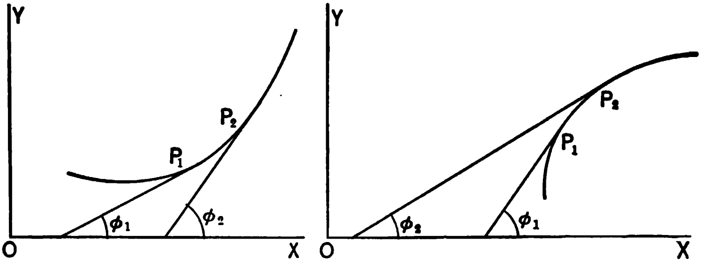
and thence $$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}, \text{ or } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

CHAPTER XVI

DIRECTION OF CURVATURE. POINTS OF INFLEXION

157. Concave Upwards or Downwards. A curve is said to be *concave upwards* at a point P, when in the immediate neighborhood of P it lies wholly above the tangent at P, as in the first figure below. Similarly, it is said to be *concave downwards*, when in the immediate neighborhood of P it lies wholly below the tangent at P, as in the second figure below.

It will now be shown that when the equation of the curve is in rectangular coördinates, the curve is concave *upwards* or *downwards*, according as $\frac{d^2y}{dx^2}$ is *positive* or *negative.*



Suppose $\frac{d^2y}{dx^2} > 0$, that is, $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$; in other words, the derivative of the slope is positive.

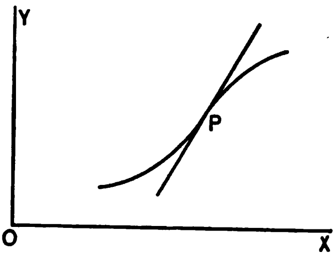
Then by Art. 21 the slope *increases* as x increases.

This case is illustrated in the first figure above, where the slope evidently *increases* as we pass from P_1 to P_2. The curve is then *concave upwards.*

But if $\frac{d^2y}{dx^2} < 0$, it follows that the slope *decreases* as x increases.

We then have the case of the second figure, where the slope *decreases* as we pass from P_1 to P_2. The curve is then *concave downwards.*

158. A Point of Inflexion is a point P where $\frac{d^2y}{dx^2}$ changes sign, the curve being concave upwards on one side of this point, and concave downwards on the other.



This can occur, provided $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are continuous, only when

$$\frac{d^2y}{dx^2}=0. \quad . \quad . \quad . \quad (1)$$

But if $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are infinite, we may have a point of inflexion

when $$\frac{d^2y}{dx^2}=\infty.$$

It is evident that the tangent at a point of inflexion crosses the curve at that point.

For example, find the point of inflexion of the curve

$$2y=2-8x+6x^2-x^3.$$

Here $$\frac{d^2y}{dx^2}=3(2-x).$$

Putting this equal to zero, we have for the required point of inflexion, $x=2$. If $x<2$, $\frac{d^2y}{dx^2}>0$; and if $x>2$, $\frac{d^2y}{dx^2}<0$.

Hence the curve is concave upwards on the left, and concave downwards on the right, of the point of inflexion.

EXAMPLES

Find the points of inflexion and the direction of curvature of the five following curves:

1. $y=(x^2-1)^2$.

Ans. $x=\pm\frac{1}{\sqrt{3}}$; concave downwards between these points, concave upwards elsewhere.

2. $y=x^4-16x^3+42x^2-28x$.

Ans. $x=1$ and $x=7$; concave downwards between these points, concave upwards elsewhere.

3. $a^4y=x(x-a)^4+a^4x$.

Ans. $x=\frac{2a}{5}$; concave downwards on the left of this point, concave upwards on the right.

4. The witch (Art. 126), $y=\frac{8a^3}{x^2+4a^2}$.

Ans. $\left(\pm\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$; concave downwards between these points, concave upwards outside of them.

5. The curve, $y=\frac{x^3}{x^2+3a^2}$.

Ans. $\left(-3a, -\frac{9a}{4}\right)$, $(0, 0)$, $\left(3a, \frac{9a}{4}\right)$; concave upwards on the left of first point, downwards between first and second, upwards between second and third, and downwards on the right of third point.

Find the points of inflexion of the following curves:

6. $y = \frac{4x}{x^2+4}$. *Ans.* $x = 0$ and $\pm 2\sqrt{3}$.

7. $y = \frac{a^2x}{(x-a)^2}$. *Ans.* $x = -2a$.

8. $y = (x^2+x)e^{-x}$. *Ans.* $x = 0$ and $x = 3$.

9. $y = e^{-ax} - e^{-bx}$. *Ans.* $x = \frac{2(\log a - \log b)}{a-b}$.

10. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ (Art. 133). *Ans.* $x = \pm\frac{a}{\sqrt{2}}$.

11. $a^4y^2 = a^2x^4 - x^6$ (Art. 134). *Ans.* $x = \pm\frac{a}{6}\sqrt{27 - 3\sqrt{33}}$.

CHAPTER XVII

CURVATURE. RADIUS OF CURVATURE. EVOLUTE AND INVOLUTE

159. Curvature. If a point moves in a straight line, the direction of its motion is the same at every point of its course, but if its path is a curved line, there is a continual change of direction as it moves along the curve. This change of direction is called *curvature.*

We have seen in the preceding chapter that the sign of the second derivative shows which way the curve bends. We shall now find that the first and second derivatives give an exact measure of the curvature.

The direction at any point being the same as that of the tangent at that point, the curvature may be measured by comparing the linear motion of the point with the simultaneous angular motion of the tangent.

160. Uniform Curvature. The curvature is uniform when, as the point moves over equal arcs, the tangent turns through equal angles. The only curve of uniform curvature is the circle. Here the measure of curvature is the ratio between the angle described by the tangent and the arc described by the point of contact. In other words, it is *the angle described by the tangent while the point describes a unit of arc.*

Suppose the point P to move in the circle AQ.

Let s denote its distance AP from some initial position A, and ϕ the angle PTX made by the tangent PT with OX.

Then as the point moves from P to Q, s is increased by $PQ = \Delta s$, and ϕ by the angle $QRK = \Delta\phi$.

As the point describes the arc Δs, the tangent turns through the angle $\Delta\phi$.

The curvature, being uniform, is then equal to $\frac{\Delta\phi}{\Delta s}$.

If we draw the radii CP, CQ, and let r denote the radius, then

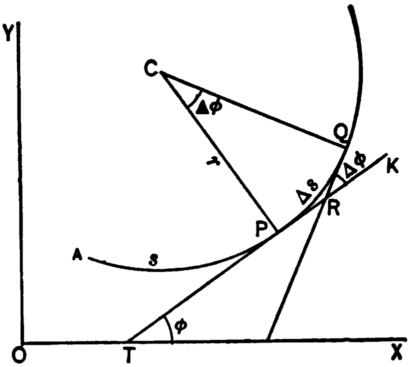
$$\text{angle } PCQ = QRK = \Delta\phi.$$

But

$$\text{arc } PQ = CP \text{ (angle } PCQ);$$

that is,

$$\Delta s = r\Delta\phi, \quad \frac{\Delta\phi}{\Delta s} = \frac{1}{r}.$$



Hence *the curvature of a circle is the reciprocal of its radius.*

For example, suppose the radius of a circle to be 50 feet.

Then its curvature is $$\frac{\Delta\phi}{\Delta s} = \frac{1}{50},$$

where $\Delta\phi$ is in circular measure, and Δs in feet.

In other words, for every foot of arc, the change of direction is

$$\frac{1}{50} \text{ in circular measure} = 1^\circ\, 8'\, 45''.$$

161. Variable Curvature. For all curves except the circle the curvature varies as we move along the curve. In moving over the arc Δs, $\frac{\Delta\phi}{\Delta s}$ is the mean curvature throughout the arc. The curvature at the beginning of this arc is more nearly equal to $\frac{\Delta\phi}{\Delta s}$, the shorter we take Δs.

Hence the curvature at any point of a curve is equal to

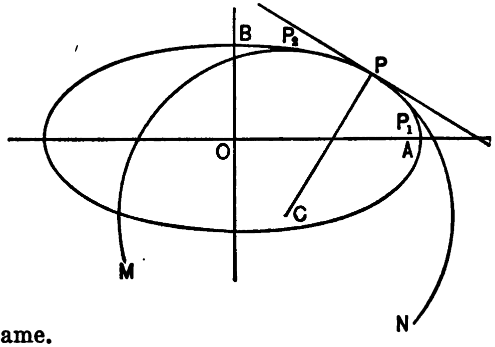
$$\text{Lim}_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}.$$

162. Circle of Curvature. A circle tangent to a curve at any point, having its concavity turned in the same direction, and having the same curvature as that of the curve at that point, is called the *circle of curvature;* its radius, the *radius of curvature;* and its centre, the *centre of curvature.*

The figure shows the circle of curvature MPN for the point P of the ellipse. C is the centre of curvature, and CP the radius of curvature.

It is to be noticed that the circle of curvature crosses the curve at P. This can be easily proved.

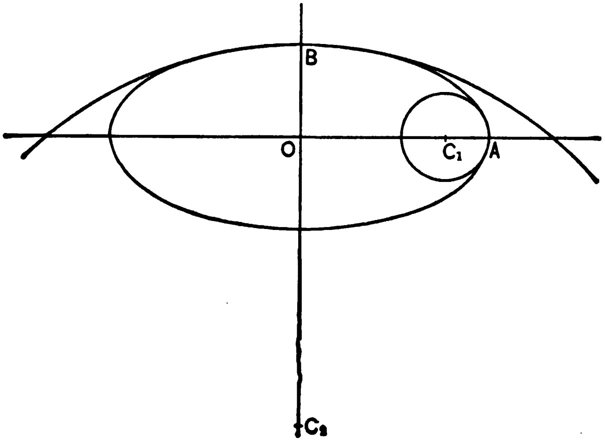
At P the circle and ellipse have the same curvature, but as we go towards P_1, the curvature of the ellipse increases, while that of the circle continues the same.



Hence on the right of P the circle is outside of the ellipse.

Moving from P to P_2, the curvature of the ellipse decreases, and therefore on the left of P the circle is inside of the ellipse.

So in general *the circle of curvature crosses the curve at the point of contact.*



The only exceptions to this rule are at points of maximum and minimum curvature, as the vertices A and B of the ellipse.

As we move from A along the curve in either direction, the curvature of the ellipse decreases; hence the circle of curvature at A lies entirely within the ellipse.

Similarly it appears that the circle of curvature at B lies entirely without the ellipse.

163. Radius of Curvature. The curvature of the circle of curvature being that of the given curve, is equal to $\frac{d\phi}{ds}$ (Art. 161). If we denote the radius of curvature by ρ, then by Art. 160,

$$\rho = \frac{ds}{d\phi}. \quad \dots \dots \dots \quad (1)$$

To obtain ρ in terms of x and y, we may write (1), $\rho = \frac{ds}{d\phi} = \frac{\frac{ds}{dx}}{\frac{d\phi}{dx}}$.

From (1) Art. 155, $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Also, $\tan\phi = \frac{dy}{dx}$, $\phi = \tan^{-1}\left(\frac{dy}{dx}\right)$.

Differentiating,

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}. \quad \dots \dots \dots \quad (2)$$

Hence

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \dots \dots \dots \quad (3)$$

It is to be noticed that ρ is always to be considered positive; that is, the sign of $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$ is taken the same as that of $\frac{d^2y}{dx^2}$.

By interchanging x and y, we have

$$\rho=\frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}},$$

which is sometimes the more convenient expression.

As an example, find the radius of curvature of the semicubical parabola $ay^2=x^3$ (Art. 130).

Differentiating, $\frac{dy}{dx}=\frac{3\,x^{\frac{1}{2}}}{2\,a^{\frac{1}{2}}}$, $\frac{d^2y}{dx^2}=\frac{3}{4\,(ax)^{\frac{1}{2}}}$.

Substituting in (3), we find

$$\rho=\frac{x^{\frac{1}{2}}(4\,a+9\,x)^{\frac{3}{2}}}{6\,a}.$$

164. Radius of Curvature in Polar Coördinates. Resuming (1), Art. 163, $\rho=\frac{ds}{d\phi}$, let us express ρ in terms of r and θ.

We may write $\rho=\frac{ds}{d\phi}=\frac{\frac{ds}{d\theta}}{\frac{d\phi}{d\theta}}$.

From (3), Art. 156, $\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$.

From (2), Art. 152,

$$\phi=\theta+\psi,\quad \therefore\ \frac{d\phi}{d\theta}=1+\frac{d\psi}{d\theta}.$$

From (1), Art. 152,

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}},\quad \psi = \tan^{-1}\left[\frac{r}{\frac{dr}{d\theta}}\right].$$

Differentiating,
$$\frac{d\psi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

Substituting,
$$\frac{d\phi}{d\theta} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

Hence
$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}. \quad \ldots\ldots \quad (1)$$

EXAMPLES

Find the radius of curvature of the following curves:

1. $y = (x-1)^2(x-2)$, at (1, 0) and (2, 0). *Ans.* $\rho = \frac{1}{2}$ and $\frac{1}{\sqrt{2}}$

2. $y = \log x$, when $x = \frac{3}{4}$. *Ans.* $\rho = 2\frac{29}{48}$.

3. The cubical parabola (Art. 130), $a^2y = x^3$. *Ans.* $\rho = \frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^4x}$.

4. The parabola, $y^2 = 4ax$. *Ans.* $\rho = \frac{2(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$.

Find the point of the parabola where $\rho = 54a$. *Ans.* $x = 8a$.

5 The equilateral hyperbola, $2xy = a^2$. *Ans.* $\rho = \frac{(x^2 + y^2)^{\frac{3}{2}}}{a^2}$.

6. The ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *Ans.* $\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$.

What are the values of ρ at the extremities of the axes?

Ans. $\frac{b^2}{a}$ and $\frac{a^2}{b}$.

7. Show that the radius of curvature of the curve,

$x^2+y^2+10x-4y+20=0$ is constant, and equal to 3.

Find the radius of curvature of the following curves:

8. $y+\log(1-x^2)=0$. *Ans.* $\rho=\frac{(1+x^2)^2}{2(1-x^2)}$.

9. $\sin y=e^x$. *Ans.* $\rho=e^{-x}$.

10. The catenary (Art. 128), $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. *Ans.* $\rho=\frac{y^2}{a}$.

11. The hypocycloid (Art. 132), $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $\rho=3(axy)^{\frac{1}{3}}$.

12. The curve $a^4y^2=a^2x^4-x^6$ (Art. 133), at the points (0, 0) and $(a, 0)$. *Ans.* $\rho=\frac{a}{2}$ and $\rho=a$.

13. The cycloid, $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$ (Art. 284).

Ans. $\rho=4a\sin\frac{\phi}{2}$

14. Show that the radius of curvature of the logarithmic spiral (Art. 138), $r=e^{a\theta}$, is proportional to r. $\rho=r\sqrt{1+a^2}$.

15. Show that the radius of curvature of the curve (Art. 135), $r=a\sin\theta+b\cos\theta$, is constant. $\rho=\frac{1}{2}\sqrt{a^2+b^2}$.

16. The spiral of Archimedes (Art. 136), $r=a\theta$.

Ans. $\rho=\frac{(r^2+a^2)^{\frac{3}{2}}}{r^2+2a^2}$.

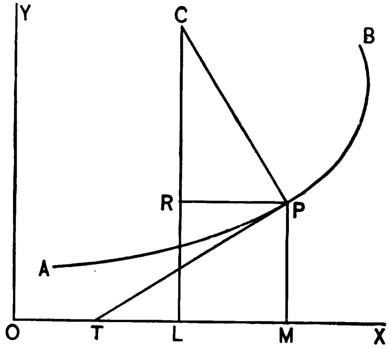
17. The cardioid (Art. 141), $r = a\,(1 - \cos\theta)$. *Ans.* $\rho^2 = \frac{8}{9}ar.$

18. The curve, $r = a\sin^3\frac{\theta}{3}$ (Art. 145). *Ans.* $\rho = \frac{3}{4}a\sin^2\frac{\theta}{3}.$

19. The parabola (Art. 139), $r = a\sec^2\frac{\theta}{2}$. *Ans.* $\rho = 2\,a\sec^3\frac{\theta}{2}.$

20. The lemniscate (Art. 143), $r^2 = a^2\cos 2\,\theta$. *Ans.* $\rho = \frac{a^2}{3r}.$

165. Coördinates of the Centre of Curvature. Let x, y be the coördinates of P, any point of the curve AB, and C the corresponding centre of curvature. CP is then the radius of curvature, and is normal to the curve. Draw also the tangent PT.



Then $CP = \rho$;

angle $PCR = PTX = \phi$.

Let α, β, be the coördinates of C. $OL = OM - RP$,

$LC = MP + RC$;

that is,

$$\alpha = x - \rho\sin\phi,$$
$$\beta = y + \rho\cos\phi. \qquad (1)$$

To express α and β in terms of x and y, we have, by (2), Art. 155, and (1), (2), Art. 163,

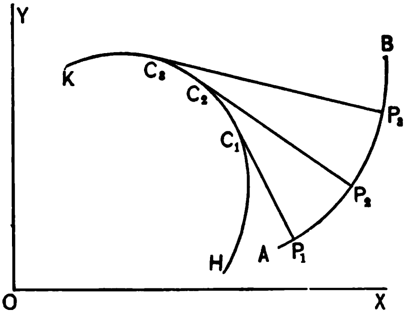
$$\rho\sin\phi = \frac{ds}{d\phi}\frac{dy}{ds} = \frac{dy}{d\phi} = \frac{dy}{dx}\frac{dx}{d\phi} = \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}},$$

$$\rho\cos\phi = \frac{ds}{d\phi}\frac{dx}{ds} = \frac{dx}{d\phi} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

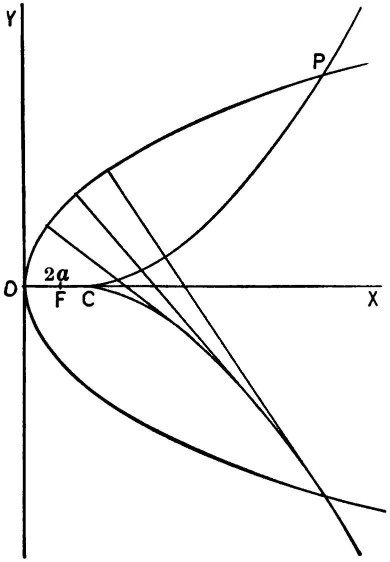
Hence

$$\alpha = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}, \quad \beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \quad \cdots\cdots \quad (2)$$

166. Evolute and Involute. Every point of a curve AB has a corresponding centre of curvature. Thus, P_1, P_2, P_3, etc., have for their respective centres of curvature C_1, C_2, C_3, etc. The curve HK, which is the locus of the centres of curvature, is called the *evolute* of AB. To express the inverse relation, AB is called the *involute* of HK.



167. To find the Equation of the Evolute of a Given Curve. By (2), Art. 165, α and β, the coördinates of any point of the required evolute, may be expressed in terms of x and y, the coördinates of any point of the given curve. These two equations, together with that of the given curve, furnish three equations between α, β, x, and y, from which, if x and y are eliminated, we obtain a relation between α and β, which is the equation of the required evolute.



For example, find the equation of the evolute of the parabola

$$y^2 = 4\,ax.$$

Here $$\frac{dy}{dx} = a^{\frac{1}{2}}x^{-\frac{1}{2}},$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{3}{2}}.$$

Substituting in (2), Art. 165, we have

$$\alpha = 3\,x + 2\,a,$$

$$\beta = -\frac{2\,x^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

Eliminating x, we have for the equation of the evolute,

$$a\beta^2 = \frac{4}{27}(\alpha - 2a)^3.$$

This curve is the semicubical parabola (Art. 130). The figure shows its form and position. F is the focus of the given parabola.

$$OC = 2a = 2\,OF.$$

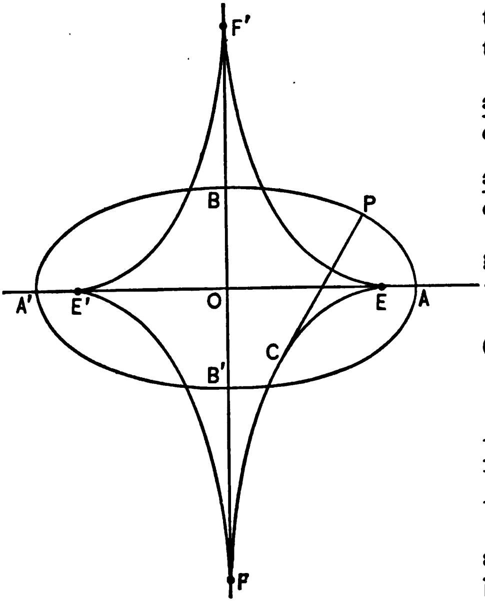
As another example, let us find the equation of the evolute of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}. \qquad \text{(Art. 66)}$$

Substituting in (2), Art. 165,

$$\alpha = \frac{(a^2 - b^2)x^3}{a^4}, \quad \beta = -\frac{(a^2 - b^2)y^3}{b^4}.$$



To eliminate x and y between these equations and that of the ellipse, we find

$$\frac{x^3}{a^3} = \frac{a\alpha}{a^2 - b^2}, \quad \frac{y^3}{b^3} = -\frac{b\beta}{a^2 - b^2},$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}}}{(a^2 - b^2)^{\frac{2}{3}}} = 1,$$

giving, for the equation of the evolute,

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

The evolute is $EF'E'FE$. E is centre of curvature for A; C for P; F for B; E' for A'; F' for B'.

In the figure F and F' are outside the ellipse, but if the eccentricity is decreased, so that $a<b\sqrt{2}$, these points fall within the ellipse.

168. Properties of the Involute and Evolute. Let us return to the equations, (1), Art. 165,

$$\alpha=x-\rho\sin\phi,$$
$$\beta=y+\rho\cos\phi.$$

Differentiating with respect to s,

$$\frac{d\alpha}{ds}=\frac{dx}{ds}-\frac{d\rho}{ds}\sin\phi-\rho\cos\phi\frac{d\phi}{ds} \quad . \quad . \quad (1)$$

$$\frac{d\beta}{ds}=\frac{dy}{ds}+\frac{d\rho}{ds}\cos\phi-\rho\sin\phi\frac{d\phi}{ds}. \quad . \quad . \quad . \quad (2)$$

Substituting in (1), $\rho=\frac{ds}{d\phi}$ and $\cos\phi=\frac{dx}{ds}$, (Art. 155), two terms cancel each other, giving

$$\frac{d\alpha}{ds}=-\frac{d\rho}{ds}\sin\phi. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly in (2), $\rho=\frac{ds}{d\phi}$ and $\sin\phi=\frac{dy}{ds}$ (Art. 155), giving

$$\frac{d\beta}{ds}=\frac{d\rho}{ds}\cos\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Dividing (4) by (3), $$\frac{d\beta}{d\alpha}=-\frac{1}{\tan\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But $\frac{d\beta}{d\alpha}$ is the slope of the tangent to the evolute at any point C_1, (see fig., Art. 166), and $\tan\phi$ the slope of the tangent to the involute at the corresponding point P_1. Since by (5) one is minus the reciprocal of the other, these tangents are perpendicular to each other. In other words, a tangent to the evolute at any point C_1 is C_1P_1, the normal to the involute at P_1.

169. Again, from (3) and (4), Art. 168,

$$\left(\frac{d\alpha}{ds}\right)^2+\left(\frac{d\beta}{ds}\right)^2=\left(\frac{d\rho}{ds}\right)^2, \text{ or } \left(\frac{ds'}{ds}\right)^2=\left(\frac{d\rho}{ds}\right)^2,$$

where s' denotes the length of the arc of the evolute measured from a fixed point. Hence,

$$\frac{ds'}{ds}=\pm\frac{d\rho}{ds},\quad \frac{d}{ds}\left(s'\pm\rho\right)=0.$$

Hence, $$s'\pm\rho=\text{a constant}, \qquad (1)$$

since, if a derivative is always zero, the function can neither increase nor decrease, but is constant.

It follows from (1) that

$$\Delta(s'\pm\rho)=0,\quad \Delta s'=\pm\Delta\rho.$$

That is, the difference between any two radii of curvature P_1C_1 P_3C_3, is equal to the corresponding included arc of the evolute C_1C_3.

170. From the two properties of Arts. 168 and 169, it follows that the involute AB may be described by the end of a string unwound from the evolute HK. From this property the word *evolute* is derived.

It will be noticed that a curve has only one evolute, but an infinite number of involutes, as may be seen by varying the length of the string which is unwound.

EXAMPLES

1. Find the coördinates of the centre of curvature of the cubical parabola (Art. 130), $a^2y=x^3$.

Ans. $\alpha=\dfrac{a^4x-9x^5}{2a^4},\ \beta=\dfrac{a^4+15x^4}{6a^2x}.$

2. Find the coördinates of centre of curvature of the semicubical parabola (Art. 130), $ay^2=x^3$.

Ans. $\alpha=-x-\dfrac{9x^2}{2a},\ \beta=4\left(x+\dfrac{a}{3}\right)\sqrt{\dfrac{x}{a}}$

3. Find the coördinates of the centre of curvature of the catenary (Art. 128), $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$.

Ans. $\alpha=x-\frac{a}{4}(e^{\frac{2x}{a}}-e^{-\frac{2x}{a}}),\ \beta=2y.$

4. Show that in the parabola (Art. 129), $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$, we have the relation $\alpha+\beta=3(x+y)$.

5. Find the coördinates of the centre of curvature, and the equation of the evolute, of the hypocycloid (Art. 132), $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Ans. $\alpha=x+3x^{\frac{1}{3}}y^{\frac{2}{3}},\ \beta=y+3x^{\frac{2}{3}}y^{\frac{1}{3}},$

$$(\alpha+\beta)^{\frac{2}{3}}+(\alpha-\beta)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

6. Given the equation of the equilateral hyperbola $2xy=a^2$, show that

$$\alpha+\beta=\frac{(y+x)^3}{a^2},\quad \alpha-\beta=\frac{(y-x)^3}{a^2}.$$

Thence derive the equation of the evolute,

$$(\alpha+\beta)^{\frac{2}{3}}-(\alpha-\beta)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

7. Find the equation of the evolute of the cissoid (Art. 125),

$$y^2=\frac{x^3}{2a-x}.$$

Ans. $4096a^3\alpha+1152a^2\beta^2+27\beta^4=0$

CHAPTER XVIII

ORDER OF CONTACT. OSCULATING CIRCLE

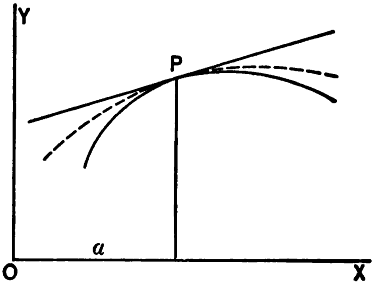
171. Order of Contact. Let us consider two curves whose equations are

$$y = \phi(x) \quad \text{and} \quad y = \psi(x).$$

If for a definite value a, of x, the value of y is the same for both curves, that is, if

$$\phi(a) = \psi(a),$$

the curves have a common point P.



If, moreover, for $x = a$, the value of $\frac{dy}{dx}$ also is the same for both curves, that is, if

$$\phi(a) = \psi(a) \text{ and } \phi'(a) = \psi'(a),$$

the curves have a common tangent at P.

The curves are then said to have a contact of the *first* order.

If besides, for $x = a$, the values of $\frac{d^2y}{dx^2}$ are the same for both curves, that is, if

$$\phi(a) = \psi(a), \quad \phi'(a) = \psi'(a), \quad \text{and} \quad \phi''(a) = \psi''(a),$$

the curves have contact of the *second order*.

In general, the conditions for a contact of the nth order at the point $x = a$, are

$$\phi(a) = \psi(a), \quad \phi'(a) = \psi'(a), \quad \phi''(a) = \psi''(a), \quad \cdots, \quad \phi^n(a) = \psi^n(a),$$

and $\phi^{n+1}(a) \neq \psi^{n+1}(a)$.

In other words, for $x = a$,

$$y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \dots,\ \frac{d^ny}{dx^n},$$

must all have the same values, respectively, taken from the equations of both curves; and $\frac{d^{n+1}y}{dx^{n+1}}$ must have different values.

172. When the Order of Contact is Even, the Curves cross at the Point of Contact; but when the Order is Odd, they do not cross. Let us distinguish the ordinates of the two curves by

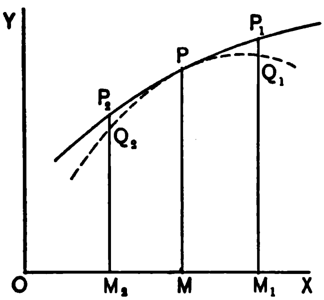
$$Y = \phi(x), \quad \text{and} \quad y = \psi(x).$$

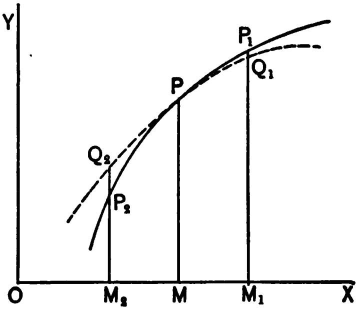
In the figures Y refers to the full curve, and y to the dotted curve.

If $Y - y$ has the same sign on both sides of P, as in the first figure, the curves do not cross at P; but if $Y - y$ is positive on one side of P and negative on the other, the curves do cross at P.

Let $\qquad OM = a, \quad MM_1 = h.$

Then $\qquad P_1Q_1 = Y - y = \phi(a + h) - \psi(a + h).$





Expanding by Taylor's Theorem,

$$P_1Q_1 = \phi(a) + h\phi'(a) + \frac{h^2}{2}\phi''(a) + \frac{h^3}{\underline{|3}}\phi'''(a) + \cdots$$

$$- \psi(a) - h\psi'(a) - \frac{h^2}{2}\psi''(a) - \frac{h^3}{\underline{|3}}\psi'''(a) - \quad \cdots. \quad \cdot \quad (1)$$

Suppose the contact of the first order; then

$$\phi(a)=\psi(a), \quad \phi'(a)=\psi'(a), \quad \text{and (1) becomes}$$

$$P_1Q_1=\frac{h^2}{\underline{|2}}\Big[\phi''(a)-\psi''(a)\Big]+\frac{h^3}{\underline{|3}}\Big[\phi'''(a)-\psi'''(a)\Big]+\cdots. \qquad (2)$$

For sufficiently small values of h the sign of the lowest power determines that of the second member, and hence the sign of P_1Q_1 will remain unchanged when $-h$ is substituted for h, giving P_2Q_2, as in the first figure.

Thus when the contact is of the first order, the curves do not cross at the point of contact.

Again, suppose the contact of the second order; then

$$\phi''(a)=\psi''(a), \text{ and (2) becomes}$$

$$P_1Q_1=\frac{h^3}{\underline{|3}}\Big[\phi'''(a)-\psi'''(a)\Big]+\frac{h^4}{\underline{|4}}\Big[\phi^{\text{iv}}(a)-\psi^{\text{iv}}(a)\Big]+\cdots.$$

Now P_1Q_1 will change sign with h, so that P_2Q_2 and P_1Q_1 will have different signs, as in the second figure.

Thus when the contact is of the second order, the curves cross at the point of contact.

By similar reasoning the general proposition is established.

It may be of service to the student, in connection with this principle, to think of two curves as having two consecutive common points, when they have contact of the first order; as having three consecutive common points, when they have contact of the second order; as having $n+1$ consecutive common points, when they have contact of the nth order.

An odd number of common points implies the crossing of the curves, but where there is an even number of common points, the curves do not cross.

173. Osculating Curves. Contact of the nth order requires that y and its first n derivatives should, for some definite value of x, have the same values for both curves.

This implies $n+1$ conditions.

The equation of the straight line, $y = ax + b$, having only two arbitrary constants, can satisfy only two of these conditions. Hence a straight line can have contact of the first order with a given curve, and cannot, in general, have contact of a higher order.

The equation of the circle $x^2 + y^2 + ax + by + c = 0$, having three arbitrary constants, can satisfy three of the conditions. Hence the circle may have contact of the second order with a given curve. Such a circle is called the *osculating circle.*

Similarly, the parabola, whose equation contains four constants, may have contact of the third order; and the general conic, whose equation contains five constants, may have contact of the fourth order with a given curve. These are called the *osculating parabola* and the *osculating conic.*

174. Order of Contact at Exceptional Points. Although the tangent has generally contact of the first order, it may at exceptional points of a curve have a contact of a higher order.

For example, since the tangent at a point of inflexion crosses the curve, it follows from Art 172, that the order of contact must be even. Hence at a point of inflexion the tangent has contact of at least the second order.

The osculating circle, which has generally contact of the second order, has a higher order of contact at points of maximum or minimum curvature, as, for example, the vertices of an ellipse. It is evident from the symmetry of the ellipse with reference to its vertices, that no circle tangent at these points would cross the curve at the point of contact. Hence, by Art. 172, the order of contact is odd, — at least the third.

175. To Find the Coördinates of the Centre, and Radius, of the Osculating Circle at Any Point of a Given Curve.

Let the equation of the given curve be

$$y = f(x).$$

The general equation of a circle with centre (a, b) and radius r, is

$$(x - a)^2 + (y - b)^2 = r^2. \quad \dots \dots \dots \quad (1)$$

Differentiating twice successively, we have

$$x-a+(y-b)\frac{dy}{dx}=0, \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (2)$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-b)\frac{d^2y}{dx^2}=0. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (3)$$

From (3),
$$y-b=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (4)$$

From (2),
$$x-a=\frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (5)$$

Substituting (4) and (5) in (1),

$$r^2=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3}{\left(\frac{d^2y}{dx^2}\right)^2}. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (6)$$

Hence
$$a=x-\frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}, \quad b=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \quad (7)$$

and
$$r=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (8)$$

In these expressions, x, $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$, refer to (1), the equation of the circle; but since the osculating circle by definition has contact of the second order with the given curve, these quantities will have the same values if derived from the equation of this curve $y=f(x)$, and applied to the point of contact.

By comparing (7) and (8) with the expressions for α, β, and ρ, in Arts. 163, 165, it is evident that the osculating circle is the same as the circle of curvature.

176. At a Point of Maximum or Minimum Curvature, the Osculating Circle has Contact of the Third Order.

If we regard equation (8) in the preceding article as referring to the given curve $y=f(x)$, we have as a condition for a maximum or minimum value of r,

$$\frac{dr}{dx}=0.$$

We thus obtain from (8),

$$3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2-\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}=0,$$

from which

$$\frac{d^3y}{dx^3}=\frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2}{1+\left(\frac{dy}{dx}\right)^2}. \quad \text{. . . . .} \quad (1)$$

Again, if we regard (8) as referring to the osculating circle

$$(x-a)^2+(y-b)^2=r^2,$$

we shall also have

$$\frac{dr}{dx}=0,$$

since r is constant for all points on the circle.

Thus we obtain, both for the curve and the circle, the same expression (1) for $\frac{d^3y}{dx^3}$, and since $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the second member of (1) have, at the point of contact, the same values for both curves, it follows that $\frac{d^3y}{dx^3}$ has likewise the same value. Hence the contact is of the third order.

EXAMPLES

1. Find the order of contact of the two curves,

$$y = x^3, \quad \text{and} \quad y = 3x^2 - 3x + 1.$$

By combining the two equations, the point $x = 1$, $y = 1$, is found to be common to both curves.

Differentiating the two given equations,

$$y = x^3, \qquad y = 3x^2 - 3x + 1,$$

$$\frac{dy}{dx} = 3x^2, \qquad \frac{dy}{dx} = 6x - 3,$$

$$\frac{d^2y}{dx^2} = 6x \qquad \frac{d^2y}{dx^2} = 6,$$

$$\frac{d^3y}{dx^3} = 6, \qquad \frac{d^3y}{dx^3} = 0.$$

When $x = 1$, $\frac{dy}{dx} = 3$, in both curves;

when $x = 1$, $\frac{d^2y}{dx^2} = 6$, in both curves;

but $\frac{d^3y}{dx^3}$ has different values in the two curves.

Hence the contact is of the second order.

2. Find the order of contact of the parabola, $4y = x^2$, and the straight line, $y = x - 1$. *Ans.* First order.

3. Find the order of contact of

$$9y = x^3 - 3x^2 + 27, \text{ and } 9y + 3x = 28.$$

Ans. Second order

4. Find the order of contact of the curves

$$y = \log(x-1), \text{ and } x^2 - 6x + 2y + 8 = 0,$$

at the common point (2, 0). *Ans.* Second order.

5. Find the order of contact of the parabola, $4y = x^2 - 4$, and the circle, $x^2 + y^2 - 2y = 3$. *Ans.* Third order.

6. What must be the value of a, in order that the parabola,

$$y = x + 1 + a(x-1)^2,$$

may have contact of the second order with the hyperbola,

$$xy = 3x - 1\,?$$

Ans. $a = -1$.

7. Find the order of contact of the parabola,

$$(x-2a)^2 + (y-2a)^2 = 2xy,$$

and the hyperbola, $xy = a^2$. *Ans.* Third order.

CHAPTER XIX

ENVELOPES

177. Series of Curves. When, in the equation of a curve, different values are assigned to one of its constants, the resulting equations represent a series of curves, differing in position, but all of the same kind or family.

For example, if we give different values to a in the equation of the parabola $y^2 = 4ax$, we obtain a series of parabolas, all having a common vertex and axis, but different focal distances.

Again, take the equation of the circle $(x-a)^2 + (y-b)^2 = c^2$. By giving different values to a, we have a series of equal circles whose centres are on the line $y = b$.

The quantity a which remains constant for any one curve of the series, but varies as we pass from one curve to another, is called the *parameter* of the series.

Sometimes two parameters are supposed to vary simultaneously, so as to satisfy a given relation between them.

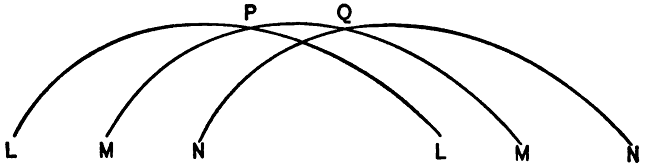
Thus, in the equation of the circle $(x-a)^2 + (y-b)^2 = c^2$, we may suppose a and b to vary, subject to the condition,

$$a^2 + b^2 = k^2.$$

We then have a series of equal circles, whose centres are on another circle described about the origin with radius k.

178. Definition of Envelope. The intersection of any two curves of a series will approach a certain limit, as the two curves approach coincidence. Now, if we suppose the parameter to vary by infinitesimal increments, the locus of the ultimate intersections of consecutive curves is called the *envelope* of the series.

179. The Envelope of a Series of Curves is Tangent to Every Curve of the Series.



Suppose L, M, N to be any three curves of the series. P is the intersection of M with the preceding curve L, and Q its intersection with the following curve N.

As the curves approach coincidence, P and Q will ultimately be two consecutive points of the envelope and of the curve M. Hence the envelope touches M.

Similarly, it may be shown that the envelope touches any other curve of the series.

180. To find the Equation of the Envelope of a Given Series of Curves.

Before considering the general problem let us take the following special example.

Required the envelope of the series of straight lines represented by

$$y = ax + \frac{m}{a},$$

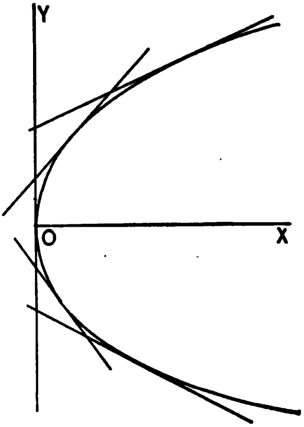
a being the variable parameter.

Let the equations of any two of these lines be

$$y = ax + \frac{m}{a}, \quad . \quad . \quad . \quad . \quad (1)$$

and $$y = (a + h)x + \frac{m}{a + h}. \quad . \quad . \quad (2)$$

From (1) and (2) as simultaneous equations, we can find the intersection of the two lines. Subtracting (1) from (2),



$$0 = hx - \frac{hm}{a(a+h)},$$

or

$$0 = x - \frac{m}{a(a+h)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (3) and (1), we have

$$x = \frac{m}{a(a+h)}, \quad y = \frac{(2a+h)m}{a(a+h)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which are the coördinates of the intersection.

Now if we suppose h to approach zero in (4), we have for the ultimate intersection of consecutive lines

$$x = \frac{m}{a^2}, \quad y = \frac{2m}{a}.$$

By eliminating a between these equations we have

$$y^2 = 4\,mx,$$

which, being independent of a, is the equation of the locus of the intersection of *any* two consecutive lines, that is, the equation of the required envelope.

The figure shows the straight lines, and the envelope, which is a parabola.

181. We will now give the general solution.

Let the given equation be

$$f(x, y, a) = 0,$$

which, by varying the parameter a, represents the series of curves.

To find the intersection of any two curves of the series, we combine

$$f(x, y, a) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$f(x, y, a+h) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2), we have

$$\frac{f(x, y, a+h) - f(x, y, a)}{h} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and it is evident that the intersection may be found by combining (1) and (3), instead of (1) and (2).

When the two curves approach coincidence, h approaches zero, and we have, by Art. 15, for the limit of equation (3),

$$\frac{\partial}{\partial a} f(x, y, a) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus equations (1) and (4) determine the intersection of two consecutive curves. By eliminating a between (1) and (4) we shall obtain the equation of the locus of these ultimate intersections, which is the equation of the envelope.

182. Applying this method to the preceding example,

$$y = ax + \frac{m}{a},$$

we differentiate with respect to a, and obtain for (4) Art. 181,

$$0 = x - \frac{m}{a^2}.$$

Eliminating a between these equations gives the equation of the envelope,

$$y^2 = 4\,mx, \quad \text{as found in Art. 180.}$$

183. The Evolute of a Given Curve is the Envelope of its Normals. This is indicated by the figure of Art. 166, and the proposition may be proved by the method of Art 181, as follows:

The general equation of the normal at the point (x', y') is by (3), Art. 148,

$$x - x' + \frac{dy'}{dx'}(y - y') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which the variable parameter is x', the quantities y', $\frac{dy'}{dx'}$, being functions of x'. Differentiating (1) with respect to x', we have

$$-1-\left(\frac{dy'}{dx'}\right)^2+(y-y')\frac{d^2y'}{dx'^2}=0. \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we find for the intersection of consecutive normals,

$$y=y'+\frac{1+\left(\frac{dy'}{dx'}\right)^2}{\frac{d^2y'}{dx'^2}},$$

$$x=x'-\frac{\frac{dy'}{dx'}\left[1+\left(\frac{dy'}{dx'}\right)^2\right]}{\frac{d^2y'}{dx'^2}}.$$

As these expressions are identical with the coördinates of the centre of curvature in Art. 165, it follows that the envelope of the normals coincides with the evolute.

EXAMPLES

1. Find the envelope of the series of straight lines represented by $y=2mx+m^4$, m being the variable parameter.

Differentiating the given equation with reference to m,

$$0=2x+4m^3.$$

Eliminating m between the two equations, we have for the envelope,

$$16y^3+27x^4=0.$$

2. Find the envelope of the series of parabolas
$y^2=a(x-a)$, a being the variable parameter. *Ans.* $4y^2=x^2$.

3. Find the envelope of a series of circles whose centres are on the axis of X, and radii proportional to (m times) their distance from the origin. *Ans.* $y^2=m^2(x^2+y^2)$.

4. Find the evolute of the parabola $y^2 = 4ax$ according to Art. 183, taking the equation of the normal in the form

$$y = m(x - 2a) - am^3. \qquad \textit{Ans.}\ 27ay^2 = 4(x - 2a)^3.$$

5. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, taking the equation of the normal in the form

$$by = ax \tan\phi - (a^2 - b^2)\sin\phi,$$

where ϕ is the eccentric angle.

$$\textit{Ans.}\ (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

6. Find the envelope of the straight lines represented by

$$x\cos 3\theta + y\sin 3\theta = a(\cos 2\theta)^{\frac{3}{2}},$$

θ being the variable parameter.

$$\textit{Ans.}\ (x^2 + y^2)^2 = a^2(x^2 - y^2), \text{ the lemniscate.}$$

7. Find the envelope of the series of ellipses, whose axes coincide and whose area is constant.

The equation of the ellipses is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a and b being variable parameters, subject to the condition

$$ab = k^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

calling the constant area πk^2.

Substituting in (1) the value of b from (2),

$$\frac{x^2}{a^2} + \frac{a^2y^2}{k^4} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which a is the only variable parameter. Differentiating (3) with respect to a, we have

$$-\frac{2x^2}{a^3} + \frac{2ay^2}{k^4} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating a between (3) and (4), we have

$$4x^2y^2 = k^4.$$

Second Solution. Differentiate (1), regarding both a and b as variable.

$$\frac{x^2\,da}{a^3}+\frac{y^2\,db}{b^3}=0. \qquad (5)$$

Differentiating (2) also, we have

$$b\,da+a\,db=0. \qquad (6)$$

From (5) and (6), we have

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}. \qquad (7)$$

From (7) and (1),

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{1}{2}. \qquad (8)$$

Substituting (8) in (2),

$$4\,x^2y^2=k^4.$$

8. Find the envelope of the circles whose diameters are the double ordinates of the parabola $y^2=4\,ax$. *Ans.* $y^2=4\,a\,(a+x)$.

9. Find the envelope of the straight lines $\frac{x}{a}+\frac{y}{b}=1$,

when $a^n+b^n=k^n$.

Ans. $x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}=k^{\frac{n}{n+1}}$.

10. Find the envelope of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

when $a+b=k$. *Ans.* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}$.

11. Find the envelope of the circles passing through the origin, whose centres are on the parabola $y^2=4\,ax$.

Ans. $(x+2\,a)y^2+x^3=0$.

12. Find the envelope of circles described on the central radii of an ellipse as diameters, the equation of the ellipse being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \textit{Ans.}\ (x^2+y^2)^2=a^2x^2+b^2y^2.$$

13. Find the envelope of the ellipses whose axes coincide, and such that the distance between the extremities of the major and minor axes is constant and equal to k.

Ans. A square whose sides are $(x \pm y)^2 = k^2$.

INTEGRAL CALCULUS

CHAPTER XX

INTEGRATION. STANDARD FORMS

184. Definition of Integration. The operation inverse to differentiation is called *integration.* By differentiation we find the differential of a given function, and by integration we find the function corresponding to a given differential. This function is called the *integral* of the differential.

For instance,

since $2xdx$ is the differential of x^2,

therefore x^2 is the integral of $2xdx$.

The symbol $\int$ is used to denote the integral of the expression following it.

Thus the foregoing relations would be written,

$$d(x^2) = 2xdx, \int 2xdx = x^2.$$

It is evidently the same thing, whether we consider this integral as the function whose differential is $2xdx$, or the function whose derivative is $2x$.

As regards notation, however, it is customary to write

$$\int 2xdx = x^2, \quad \text{and not} \int 2x = x^2.$$

In other words,

$$\int \text{ is the inverse of } d, \text{ and not of } \frac{d}{dx}.$$

Thus the general definition of $\int \phi(x)dx$ is that function whose differential is $\phi(x)dx$; the symbol $\int$ denoting "the function whose differential is," in the same way that the inverse symbol, $\tan^{-1}$, denotes "the angle whose tangent is."

Integration is not like differentiation a direct operation, but consists in recognizing the given expression as the differential of a known function, or in reducing it to a form where such recognition is possible.

185. Elementary Principles.

(*a*) It is evident that we may write

$$\int 2\,x\,dx = x^2 + 2, \text{ or } \int 2\,x\,dx = x^2 - 5,$$

as well as

$$\int 2\,x\,dx = x^2;$$

since the differential of $x^2 + 2$, as well as of $x^2 - 5$ is $2\,x\,dx$.

In general

$$\int 2\,x\,dx = x^2 + C,$$

where C denotes an arbitrary constant called the *constant of integration.*

Every integral in its most general form includes this term, $+ C$.

(*b*) Since

$$d(u \pm v \pm w) = du \pm dv \pm dw,$$

it follows that

$$\int (du \pm dv \pm dw) = \int du \pm \int dv \pm \int dw.$$

That is, we integrate a polynomial by integrating the separate terms, and retaining the signs.

(*c*) Since $$d(au) = a\,du,$$

it follows that $$\int a\,du = a\int du.$$

That is, a constant factor may be transferred from one side of the symbol $\int$ to the other, without affecting the integral.

186. Fundamental Integrals. Since integration is the inverse of differentiation, to integrate any given function we must reduce it to one or more of the differentials of the elementary functions, expressed by the fundamental formulæ of the Differential Calculus. Corresponding to these formulæ we may write a list of integrals, which may be regarded as fundamental, and to which all integrals should, if possible, be ultimately reduced. We shall then consider in this chapter such examples as are integrable by these formulæ, either directly, or after some simple transformation.

I. $$\int u^n du = \frac{u^{n+1}}{n+1}.$$

II. $$\int \frac{du}{u} = \log u.$$

III. $$\int a^u du = \frac{a^u}{\log a}.$$

IV. $$\int e^u du = e^u.$$

V. $$\int \cos u\,du = \sin u.$$

VI. $$\int \sin u\,du = -\cos u.$$

$$\text{VII.}\quad \int \sec^2 u\, du = \tan u.$$

$$\text{VIII.}\quad \int \operatorname{cosec}^2 u\, du = -\cot u.$$

$$\text{IX.}\quad \int \sec u \tan u\, du = \sec u.$$

$$\text{X.}\quad \int \operatorname{cosec} u \cot u\, du = -\operatorname{cosec} u.$$

$$\text{XI.}\quad \int \tan u\, du = \log \sec u.$$

$$\text{XII.}\quad \int \cot u\, du = \log \sin u.$$

$$\text{XIII.}\quad \int \sec u\, du = \log(\sec u + \tan u) = \log \tan\left(\frac{\pi}{4} + \frac{u}{2}\right).$$

$$\text{XIV.}\quad \int \operatorname{cosec} u\, du = \log(\operatorname{cosec} u - \cot u) = \log \tan \frac{u}{2}.$$

$$\text{XV.}\quad \int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a},\ \text{or} = -\frac{1}{a}\cot^{-1}\frac{u}{a}.$$

$$\text{XVI.}\quad \int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{u-a}{u+a},\ \text{or} = \frac{1}{2a}\log\frac{a-u}{a+u}.$$ [*]

$$\text{XVII.}\quad \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a},\ \text{or} = -\cos^{-1}\frac{u}{a}.$$

$$\text{XVIII.}\quad \int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right).$$ [*]

$$\text{XIX.}\quad \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a},\ \text{or} = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{u}{a}.$$

$$\text{XX.}\quad \int \frac{du}{\sqrt{2au - u^2}} = \operatorname{vers}^{-1}\frac{u}{a}.$$

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[*] For XVI. and XVIII. expressed as inverse hyperbolic functions, see Art. 281.

INTEGRALS BY I. AND II.

187. Proof of I. and II.

To derive I.,

since $$d(u^{n+1}) = (n+1)u^n\,du,$$

therefore

$$u^{n+1} = \int (n+1)u^n\,du = (n+1)\int u^n\,du, \quad \text{by (c), Art. 185.}$$

Hence $$\int u^n\,du = \frac{u^{n+1}}{n+1}.$$

Formula II. follows directly from

$$d\log u = \frac{du}{u}.$$

It is to be noticed that I. applies to all values of n except $n = -1$ For this value it gives

$$\int u^{-1}\,du = \frac{u^0}{0} = \infty.$$

Formula II. provides for this failing case of I.

EXAMPLES

Integrate the following expressions:

1. $\int x^4 dx.$

If we apply I., calling $u = x$, and $n = 4$; then $du = dx$. Then we have

$\int x^4 dx = \frac{x^5}{5} + C$, adding the constant of integration C, according to (*a*), Art. 185.

2. $\int (x^2+1)^{\frac{1}{2}} x\,dx.$

If we apply I., calling $u = x^2 + 1$, and $n = \frac{1}{2}$; then $du = 2\,x\,dx.$

We must then introduce a factor 2 before the $x\,dx$, and consequently its reciprocal $\frac{1}{2}$ on the left of $\int$

$$\int (x^2+1)^{\frac{1}{2}} x\,dx = \frac{1}{2}\int (x^2+1)^{\frac{1}{2}}\, 2\,x\,dx, \qquad \text{by } (c),\text{ Art. 185.}$$

$$= \frac{1}{2}\frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{(x^2+1)^{\frac{3}{2}}}{3} + C.$$

3. $$\int \frac{(x^2 - a^2)dx}{x^3 - 3\,a^2x} = \frac{1}{3}\int \frac{(3\,x^2 - 3\,a^2)dx}{x^3 - 3\,a^2x}$$

$$= \frac{1}{3}\log\,(x^3 - 3\,a^2x) = \log\,(x^3 - 3\,a^2x)^{\frac{1}{3}} + C.$$

By introducing the factor 3, we make the numerator the differential of the denominator, and then apply II.

4. $\int (2\,x^9 - 3\,x^6 + 12\,x^3 - 3)dx = \frac{x^{10}}{5} - \frac{3\,x^7}{7} + 3\,x^4 - 3\,x + C.$

5. $\int \left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}} + \frac{2}{x^5} - \frac{2}{x}\right)dx = \frac{3\,x^{\frac{5}{3}}}{5} - 3\,x^{\frac{1}{3}} - \frac{1}{2\,x^4} - 2\log x + C.$

6. $\int (x^2 - 2)^3 x^3\,dx = \frac{x^{10}}{10} - \frac{3\,x^8}{4} + 2\,x^6 - 2\,x^4 + C.$

7. $\int (x^2 - 2)^3 x\,dx = \frac{(x^2-2)^4}{8} + C.$

8. $\int\left(a^{\frac{1}{3}}-x^{\frac{1}{3}}\right)^3 dx = ax - \frac{9\,a^{\frac{2}{3}}x^{\frac{4}{3}}}{4} + \frac{9\,a^{\frac{1}{3}}x^{\frac{5}{3}}}{5} - \frac{x^2}{2} + C.$

9. $\int\left(x^{a-\frac{1}{2}} + a\,x^{\frac{a-b}{a+b}} - x^{\frac{1-n}{n}}\right)dx = \frac{2\,x^{a+\frac{1}{2}}}{2a+1} + \frac{(a+b)x^{\frac{2a}{a+b}}}{2} - n\,x^{\frac{1}{n}} + C.$

10. $\int\left(\sqrt[4]{\frac{x^3}{a^3}} - \sqrt[4]{\frac{a^3}{x^3}}\right)^3 dx = \frac{4\,x^{\frac{13}{4}}}{13\,a^{\frac{9}{4}}} - \frac{12\,x^{\frac{7}{4}}}{7\,a^{\frac{3}{4}}} + 12\,a^{\frac{3}{4}}x^{\frac{1}{4}} + \frac{4\,a^{\frac{9}{4}}}{5\,x^{\frac{5}{4}}} + C.$

11. $\int\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)^4 dx = \frac{3\,x^{\frac{7}{3}}}{7} + \frac{12\,x^{\frac{5}{3}}}{5} + 6\,x + 12\,x^{\frac{1}{3}} - 3\,x^{-\frac{1}{3}} + C.$

12. $\int\frac{(y-1)^3}{y}dy = \frac{y^3}{3} - \frac{3\,y^2}{2} + 3\,y - \log y + C.$

13. $\int\frac{(x+1)(x-2)}{x^{\frac{2}{3}}}dx = \frac{3\,x^{\frac{7}{3}}}{7} - \frac{3\,x^{\frac{4}{3}}}{4} - 6\,x^{\frac{1}{3}} + C.$

14. $\int(x^3+1)^5 x^2 dx = \frac{(x^3+1)^6}{18} + C.$

15. $\int(ax^2+b)^{\frac{2}{3}}x\,dx.$

16. $\int(ax+b)^n dx.$

17. $\int(ax^3+b)^n x^2\,dx.$

18. $\int(ax^n+b)^p x^{n-1}\,dx.$

19. $\int\left(1+\frac{1}{ax^2+b}\right)\frac{x\,dx}{ax^2+b}.$

20. $\int(\sqrt{w}+\sqrt{a})^4\frac{dw}{\sqrt{w}}.$

21. $\int(x^{-1}-1)^{10}\frac{dx}{x^2}.$

22. $\int(a+\log t)\frac{dt}{t}.$

23. $\int\frac{e^{x-1}+x^{e-1}}{e^x+x^e}dx.$ **24.** $\int(a-\log_{10}x)\frac{dx}{x}.$

25. $\int\left[\frac{1}{(e^{2x}+2)^2}+\frac{1}{e^{2x}-3}\right]e^{2x}\,dx.$ **26.** $\int\sin^5\theta\cos\theta\,d\theta.$

27. $\int(e^{2\theta}+\sin 2\theta)(e^{2\theta}+\cos 2\theta)d\theta.$

28. $\int\tan^6 x\sec^2 x\,dx.$ **29.** $\int\sec^5 x\tan x\,dx.$

30. $\int(\sin^m\theta+\cos^n\theta)\ \sin\theta\cos\theta\,d\theta.$

31. $\int(\sec\theta+\tan\theta)^{10}\sec\theta\,d\theta.$

32. $\int(\sin\phi+\cos\phi)^n(\sin^2\phi-\cos^2\phi)d\phi.$ **33.** $\int(a^x+b)^3 a^x\,dx.$

34. $\int\frac{\sin^{-1}x\,dx}{\sqrt{1-x^2}}.$ **35.** $\int\frac{dx}{(1+x^2)\tan^{-1}x}.$

A rational fraction, whose denominator is of the first degree, may be integrated directly or after being reduced to a mixed quantity.

36. $\int\frac{5\,dx}{4x-3}=\frac{5}{4}\log(4x-3)+C.$

37. $\int\frac{5x-1}{2x+3}dx=\frac{5x}{2}-\frac{17}{4}\log(2x+3)+C.$

38. $\int\frac{2x^3-7x^2+1}{2x-1}dx=\frac{x^3}{3}-\frac{3x^2}{2}-\frac{3x}{2}-\frac{1}{4}\log(2x-1)+C.$

39. $$\int\frac{ax+b}{bx+a}dx=\frac{ax}{b}+\frac{b^2-a^2}{b^2}\log(bx+a)+C.$$

40. $$\int\frac{x^3+a^3}{x-a}dx=\frac{x^3}{3}+\frac{ax^2}{2}+a^2x+2a^3\log(x-a)+C.$$

41. $$\int\frac{(x+a)^2}{x+b}dx=\frac{x^2}{2}+2ax-bx+(a-b)^2\log(x+b)+C.$$

42. $$\int\frac{(x+a)^3}{x-a}dx=\frac{x^3}{3}+2ax^2+7a^2x+8a^3\log(x-a)+C.$$

INTEGRALS BY III. AND IV.

188. Proof of III. and IV. These are evidently obtained directly from the corresponding formulæ of differentiation.

EXAMPLES

1. $$\int(e^{4x}+a^{5x}+3b^{-2x})\,dx=\frac{e^{4x}}{4}+\frac{a^{5x}}{5\log a}-\frac{3b^{-2x}}{2\log b}+C.$$

2. $$\int(e^{ax}+e^{-ax})^3dx=\frac{1}{a}\left[\frac{e^{3ax}}{3}+3e^{ax}-3e^{-ax}-\frac{e^{-3ax}}{3}\right]+C.$$

3. $$\int(a^{\frac{x}{2}}-a^{\frac{x}{3}})^3dx=\frac{1}{\log a}\left(\frac{2a^{\frac{3x}{2}}}{3}-\frac{9a^{\frac{4x}{3}}}{4}+\frac{18a^{\frac{7x}{6}}}{7}-a^x\right)+C.$$

4. $$\int\frac{(e^x-1)^2}{\sqrt[3]{e^{2x}}}dx=\frac{3e^{\frac{4x}{3}}}{4}-6e^{\frac{x}{3}}-\frac{3e^{-\frac{2x}{3}}}{2}+C.$$

5. $\int (e^{x+a} - e^{ax+b})\,dx.$ **6.** $\int (e^{\sin x}\cos x - a^{\cos 2x}\sin 2x)\,dx.$

7. $\int \left(e^{x^2}x + \frac{e^{\frac{1}{x}}}{x^2}\right)dx.$ **8.** $\int (e^{\tan\theta}\sec\theta - e^{\sec\theta}\tan\theta)\sec\theta\,d\theta.$

9. $\int \frac{e^{\sin 2x}\sin^2 x\,dx}{e^{2x}}.$ **10.** $\int a^x b^{2x} dx = \int (ab^2)^x dx = \frac{a^x b^{2x}}{\log (ab^2)} + C.$

11. $\int a^{mx+n} b^{px+q}\,dx = \frac{a^{mx+n}b^{px+q}}{\log (a^m b^p)} + C.$

12. $\int \frac{a^{2x} + b^{2x}}{e^x}\,dx = \frac{1}{e^x}\left(\frac{a^{2x}}{2\log a - 1} + \frac{b^{2x}}{2\log b - 1}\right) + C.$

13. $\int (ab^x - ba^x)^2\,dx = \frac{a^2 b^{2x}}{2\log b} - \frac{2\,(ab)^{x+1}}{\log (ab)} + \frac{b^2 a^{2x}}{2\log a} + C.$

INTEGRALS BY V.—XIV.

189. Proof of V.—XIV. It is evident that V.—X. are obtained directly from the corresponding formulæ of differentiation.

To derive XI. and XII.,

$$\int \tan u\,du = -\int \frac{-\sin u\,du}{\cos u} = -\log \cos u = \log \sec u.$$

$$\int \cot u\,du = \int \frac{\cos u\,du}{\sin u} = \log \sin u.$$

To derive XIII. and XIV.,

$$\int \sec u\,du = \int \frac{\sec u\,(\tan u + \sec u)\,du}{\sec u + \tan u} = \int \frac{\sec u\tan u\,du + \sec^2 u\,du}{\sec u + \tan u}$$

$$= \log(\sec u + \tan u).$$

$$\int \operatorname{cosec} u\,du = \int \frac{\operatorname{cosec} u\,(-\cot u + \operatorname{cosec} u)\,du}{\operatorname{cosec} u - \cot u}$$

$$= \log(\operatorname{cosec} u - \cot u).$$

By Trigonometry,

$$\operatorname{cosec} u - \cot u = \frac{1-\cos u}{\sin u} = \frac{2\sin^2\frac{u}{2}}{2\sin\frac{u}{2}\cos\frac{u}{2}} = \tan\frac{u}{2}.$$

If we substitute in this, $\frac{\pi}{2}+u$ for u,

we have $$\sec u + \tan u = \tan\left(\frac{\pi}{4}+\frac{u}{2}\right).$$

Thus we obtain the second forms of XIII. and XIV.

1. $$\int\left(\sin 3x + \cos 5x - \sin\frac{x}{2}\right)dx = -\frac{\cos 3x}{3} + \frac{\sin 5x}{5} + 2\cos\frac{x}{2} + C.$$

2. $$\int\left(\sin\frac{x+a}{m} + \cos\frac{x+b}{n}\right)dx = -m\cos\frac{x+a}{m} + n\sin\frac{x+b}{n} + C.$$

3. $$\int\frac{1+\sin mx}{\cos^2 mx}dx = \frac{1}{m}(\tan mx + \sec mx) + C.$$

4. $\int(\sec 5x - \tan 5x)\sec 5x\,dx.$ 5. $\int(\sec 2\theta + \tan 2\theta)\,d\theta.$

6. $\int\left(\frac{\sin\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\sin\theta}\right)d\theta.$ 7. $\int(\sin\theta - \operatorname{vers}\theta)^2\,d\theta.$

8. $$\int\frac{\operatorname{vers}^2 x}{\sin x}dx = \cos x - 2\log(1+\cos x) + C.$$

9. $\int\frac{\operatorname{vers}^2 x}{\cos^2 x}dx.$ 10. $\int\frac{\operatorname{vers}^2 x}{\sin^2 x}dx.$

11. $$\int\frac{\sec\phi\,d\phi}{a\sin\phi + b\cos\phi} = \frac{1}{a}\log(a\tan\phi + b) + C.$$

12. $\int(\tan x - \cot x + 1)^2\,dx = \tan x - \cot x - 3x - 2\log\sin 2x + C$

13. $\int(\sec 2x + \tan 2x - \cot 2x)^2\,dx = \tan 2x + \sec 2x - \frac{1}{2}\cot 2x$

$$-\log\tan x - 4x + C.$$

14. $\int(\sec\phi + \operatorname{cosec}\phi - 1)^2\,d\phi = \phi + \tan\phi - \cot\phi$

$$+ 2\log\frac{1+\cos\phi}{1+\sin\phi} + C.$$

The following may be integrated after trigonometric transformation.

15. $\int\sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$

16. $\int\cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$

17. $\int\operatorname{vers}^2 x\,dx.$ **18.** $\int\sin^2 x\cos^2 x\,dx = \frac{x}{8} - \frac{\sin 4x}{32} + C.$

19. $\int\sin m\theta\sin n\theta\,d\theta = \frac{\sin(m-n)\theta}{2(m-n)} - \frac{\sin(m+n)\theta}{2(m+n)} + C.$

20. $\int\cos m\theta\cos n\theta\,d\theta = \frac{\sin(m-n)\theta}{2(m-n)} + \frac{\sin(m+n)\theta}{2(m+n)} + C.$

21. $\int\sin m\theta\cos n\theta\,d\theta = -\frac{\cos(m-n)\theta}{2(m-n)} - \frac{\cos(m+n)\theta}{2(m+n)} + C.$

22. $\int\cos 5x\cos 2x\,dx = \frac{\sin 3x}{6} + \frac{\sin 7x}{14} + C.$

23. $\int \sin(3x+2)\cos(4x+3)\,dx = \frac{\cos(x+1)}{2} - \frac{\cos(7x+5)}{14} + C.$

24. $\int \sin x \sin 2x \sin 3x\,dx = \frac{\cos 6x}{24} - \frac{\cos 4x}{16} - \frac{\cos 2x}{8} + C.$

25. $\int \frac{\sin 3\theta}{\sin\theta}\,d\theta.$ 26. $\int \frac{\sin 4\theta}{\sin\theta}\,d\theta = 2\sin\theta + \frac{2\sin 3\theta}{3} + C.$

27. $\int \frac{d\theta}{1+\sin\theta} = \tan\theta - \sec\theta + C.$ 28. $\int \frac{d\theta}{\operatorname{vers}\theta}.$

29. $\int \frac{1-\sin x}{\operatorname{vers} x}\,dx = -\operatorname{cosec} x - \cot x - \log \operatorname{vers} x + C.$

30. $\int \sqrt{1+\sin x}\,dx = \int \frac{\cos x\,dx}{\sqrt{1-\sin x}} = -2\sqrt{1-\sin x} + C.$

31. $\int \sqrt{\operatorname{vers} x}\,dx.$ 32. $\int \frac{d\theta}{\sin\theta+\cos\theta} = \frac{1}{\sqrt{2}}\log\tan\left(\frac{\theta}{2}+\frac{\pi}{8}\right) + C.$

INTEGRALS BY XV.—XX.

190. Proof of XV.—XX.

To derive XV.,

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\int \frac{\frac{du}{a}}{1+\frac{u^2}{a^2}} = \frac{1}{a}\int \frac{d\left(\frac{u}{a}\right)}{1+\left(\frac{u}{a}\right)^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}.$$

To derive XVII.,

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{\frac{du}{a}}{\sqrt{1-\frac{u^2}{a^2}}} = \sin^{-1}\frac{u}{a}.$$

To derive XIX.,

$$\int\frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\int\frac{\frac{du}{a}}{\frac{u}{a}\sqrt{\frac{u^2}{a^2}-1}}=\frac{1}{a}\sec^{-1}\frac{u}{a}.$$

To derive XX.,

$$\int\frac{du}{\sqrt{2au-u^2}}=\int\frac{\frac{du}{a}}{\sqrt{2\frac{u}{a}-\frac{u^2}{a^2}}}=\text{vers}^{-1}\frac{u}{a}.$$

Since $$\tan^{-1}\frac{u}{a}=\frac{\pi}{2}-\cot^{-1}\frac{u}{a},$$

it is evident that $$d\tan^{-1}\frac{u}{a}=d\left(-\cot^{-1}\frac{u}{a}\right).$$

Hence either expression may be used as the integral in XV.

In the same way we obtain the second forms of XVII. and XIX.

The formulæ XVI. and XVIII. are inserted in the list of integrals, because their forms are similar to XV. and XVII., respectively, with different signs.

To derive XVI.,

$$\frac{1}{u^2-a^2}=\frac{1}{2a}\left(\frac{1}{u-a}-\frac{1}{u+a}\right);$$

hence

$$\int\frac{du}{u^2-a^2}=\frac{1}{2a}\int\left(\frac{du}{u-a}-\frac{du}{u+a}\right)$$

$$=\frac{1}{2a}[\log(u-a)-\log(u+a)]=\frac{1}{2a}\log\frac{u-a}{u+a}.$$

Or we may integrate thus:

$$\int\frac{du}{u^2-a^2}=\frac{1}{2a}\int\left(\frac{-du}{a-u}-\frac{du}{a+u}\right)$$

$$=\frac{1}{2a}[\log(a-u)-\log(a+u)]=\frac{1}{2a}\log\frac{a-u}{a+u}.$$

To derive XVIII.,

assume $\sqrt{u^2\pm a^2}=z$, a new variable.

Then $u^2\pm a^2=z^2,$

$$2udu=2zdz;$$

therefore $\frac{du}{z}=\frac{dz}{u}=\frac{du+dz}{u+z}.$

Hence $\int\frac{du}{z}=\int\frac{du+dz}{u+z}=\log(u+z);$

that is, $\int\frac{du}{\sqrt{u^2\pm a^2}}=\log(u+\sqrt{u^2\pm a^2}).$

EXAMPLES

1. $\int\frac{dx}{4x^2+9}=\frac{1}{6}\tan^{-1}\frac{2x}{3}+C.$ 2. $\int\frac{dx}{4x^2-9}=\frac{1}{12}\log\frac{2x-3}{2x+3}+C$

3. $\int\frac{dx}{\sqrt{4-25x^2}}=\frac{1}{5}\sin^{-1}\frac{5x}{2}+C.$

4. $\int\frac{dx}{\sqrt{25x^2-4}}=\frac{1}{5}\log(5x+\sqrt{25x^2-4})+C.$

5. $\int\frac{dx}{\sqrt{5x^2+1}}=\frac{1}{\sqrt{5}}\log(x\sqrt{5}+\sqrt{5x^2+1})+C.$

6. $\int \frac{dx}{3-12x^2} = \frac{1}{12} \log \frac{2x+1}{2x-1} + C.$

7. $\int \frac{dy}{12y^2+3}$. 8. $\int \frac{dw}{12w^2-3}$. 9. $\int \frac{dx}{3x^2-5}$.

10. $\int \frac{dt}{\sqrt{3t^2-2}}$. 11. $\int \frac{dx}{\sqrt{2-3x^2}}$. 12. $\int \frac{dx}{\sqrt{3x^2+2}}$

13. $\int \frac{dx}{x\sqrt{9x^2-4}} = \frac{1}{2} \sec^{-1} \frac{3x}{2} + C.$

14. $\int \frac{dx}{\sqrt{mx-x^2}} = \text{vers}^{-1} \frac{2x}{m} + C.$

15. $\int \frac{dx}{x\sqrt{a^2x^2-16}} = \frac{1}{4} \sec^{-1} \frac{ax}{4} + C.$

16. $\int \frac{dx}{\sqrt{7x-4x^2}} = \frac{1}{2} \text{vers}^{-1} \frac{8x}{7} + C.$

17. $\int \frac{dx}{x\sqrt{4-(\log x)^2}} = \sin^{-1} \log \sqrt{x} + C.$

18. $\int \frac{3x-2}{x^2+9} dx = \frac{3}{2} \log (x^2+9) - \frac{2}{3} \tan^{-1} \frac{x}{3} + C.$

19. $\int \frac{x+3}{4x^2-5} dx = \frac{1}{8} \log (4x^2-5) + \frac{3}{4\sqrt{5}} \log \frac{2x-\sqrt{5}}{2x+\sqrt{5}} + C.$

20. $\int \frac{3x-2}{\sqrt{9-x^2}} dx = -3\sqrt{9-x^2} - 2 \sin^{-1} \frac{x}{3} + C.$

21. $$\int\frac{x+3}{\sqrt{x^2+4}}\,dx=\sqrt{x^2+4}+3\log\left(x+\sqrt{x^2+4}\right)+C.$$

22. $$\int\frac{5x-1}{\sqrt{3x^2-9}}\,dx=\frac{5}{3}\sqrt{3x^2-9}-\frac{1}{\sqrt{3}}\log\left(x\sqrt{3}+\sqrt{3x^2-9}\right)+C.$$

23. $$\int\frac{d\theta}{a^2\sin^2\theta+b^2\cos^2\theta}=\frac{1}{ab}\tan^{-1}\frac{a\tan\theta}{b}+C.$$

24. $$\int\frac{d\phi}{1+\cos^2\phi}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan\phi}{\sqrt{2}}+C.$$

25. $$\int\frac{\sin\theta\,d\theta}{\sqrt{3\cos^2\theta+4\sin^2\theta}}=\cos^{-1}\left(\frac{\cos\theta}{2}\right)+C.$$

26. $$\int\frac{\sin 2x\,dx}{\sqrt{5\cos^2 2x-4\sin^2 2x}}=-\frac{1}{6}\log\left(3\cos 2x+\sqrt{9\cos^2 2x-4}\right)+C$$

27. $$\int\frac{e^{2x}+e^x}{e^{2x}+a}\,dx=\frac{1}{2}\log\left(e^{2x}+a\right)+\frac{1}{\sqrt{a}}\tan^{-1}\frac{e^x}{\sqrt{a}}+C.$$

The same formulæ may be applied to integrals involving x^2+ax+b or $-x^2+ax+b$, by completing the square with the terms containing x. Thus,

28. $$\int\frac{dx}{x^2+6x+13}=\int\frac{dx}{(x+3)^2+4}=\frac{1}{2}\tan^{-1}\frac{x+3}{2}+C.$$

29. $$\int\frac{dx}{\sqrt{8+4x-4x^2}}=\int\frac{dx}{\sqrt{9-(2x-1)^2}}=\frac{1}{2}\sin^{-1}\frac{2x-1}{3}+C.$$

30. $$\int\frac{dx}{\sqrt{3x^2-4x+2}}=\frac{1}{\sqrt{3}}\log\left(3x-2+\sqrt{9x^2-12x+6}\right)+C.$$

31. $$\int\frac{dx}{2x^2-3x+5}=\frac{2}{\sqrt{31}}\tan^{-1}\frac{4x-3}{\sqrt{31}}+C.$$

32. $\int\frac{dx}{x^2-ax+9}$, when $a=4$; when $a=6$; when $a=8$.

33. $$\int\frac{dx}{(x+a)^2+(x+b)^2}=\frac{1}{a-b}\tan^{-1}\frac{2x+a+b}{a-b}+C.$$

34. $$\int\frac{dx}{\sqrt{-4x^2+2x+1}}=\frac{1}{2}\sin^{-1}\frac{4x-1}{\sqrt{5}}+C.$$

35. $$\int\frac{dx}{(x+a)(x+b)}=\frac{1}{a-b}\log\frac{x+b}{x+a}+C.$$

36. $$\int\frac{dx}{\sqrt{(x-a)(b-x)}}=\sin^{-1}\frac{2x-a-b}{a-b}+C.$$

37. $$\int\frac{dx}{(x+a)^3-(x+b)^3}=\frac{2}{\sqrt{3}(a-b)^2}\tan^{-1}\frac{(2x+a+b)\sqrt{3}}{a-b}+C.$$

38. $$\int\frac{dx}{\sqrt{(x+a)(x+b)}}=2\log\left(\sqrt{x+a}+\sqrt{x+b}\right)+C.$$

CHAPTER XXI

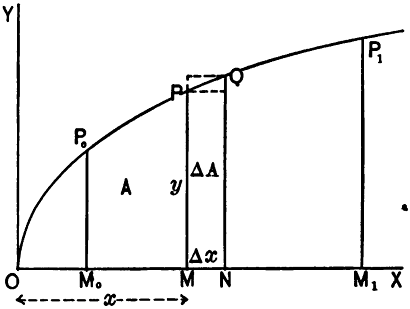
SIMPLE APPLICATIONS OF INTEGRATION. CONSTANT OF INTEGRATION

Before continuing the integration of functions, we will consider the relation of integration to the determination of the area bounded by a given curve, and show how the constant of integration may be determined.

191. Derivative of an Area. Let $y=f(x)$ be the equation of a given curve OP_1. Suppose a point to move along the curve starting from P_0, and let x, y, be the coördinates of any position P.

At the same time the ordinate of the moving point starts from the position P_0M_0, and sweeps over or generates a certain area. When the point has moved to P, this area is P_0M_0MP. Denote this area by A.

A is a function of x, and it will now be proved that its derivative with respect to x is equal to y.



Give to x the increment $\Delta x = MN$.

Then
$$\Delta A = PMNQ.$$
$$\Delta A > y\,\Delta x, \text{ and } \Delta A < (y+\Delta y)\,\Delta x,$$
$$\frac{\Delta A}{\Delta x} > y, \quad \text{and } \frac{\Delta A}{\Delta x} < y+\Delta y.$$

Hence
$$\frac{dA}{dx} = \text{Lim}_{\Delta x=0}\frac{\Delta A}{\Delta x} = y.$$

In case the curve *descends* from P to Q, the above inequalities will be reversed, but the result will be the same.

192. Area under Curve. Let it be required to find the area $P_0M_0M_1P_1$, between the curve, the axis of X, and the two ordinates P_0M_0 and P_1M_1.

Let $OM_0 = a$, and $OM_1 = b$.

From the preceding article $\frac{dA}{dx} = y$.

Hence $$A = \int y\,dx = \int f(x)dx.$$

Let $$\int f(x)dx = F(x) + C;$$

then $$A = F(x) + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To determine C, we have the condition that A begins when $x = a$; that is, $A = 0$ when $x = a$.

Hence $$0 = F(a) + C, \quad C = -F(a).$$

Substituting in (1), $A = F(x) - F(a) = P_0M_0MP$. . . . . (2)

It is to be noticed that C is determined by the initial value a of x, corresponding to the initial ordinate P_0M_0.

If now we let $x = b$ in (2), we have

$$A = F(b) - F(a) = P_0M_0M_1P_1.$$

For example, let the given curve be the parabola $y^2 = x$.

Then $$A = \int y\,dx = \int x^{\frac{1}{2}}dx = \frac{2\,x^{\frac{3}{2}}}{3} + C. \quad . \quad . \quad . \quad (3)$$

To determine C, $A = 0$ when $x = a$.

$$0 = \frac{2\,a^{\frac{3}{2}}}{3} + C, \quad C = -\frac{2\,a^{\frac{3}{2}}}{3}.$$

Substituting in (3),

$$A = \frac{2\,x^{\frac{3}{2}}}{3} - \frac{2\,a^{\frac{3}{2}}}{3} = P_0M_0MP. \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

To find $P_0M_0M_1P_1$, let $x = b$ in (4).

$$A = \frac{2\,b^{\frac{3}{2}}}{3} - \frac{2\,a^{\frac{3}{2}}}{3}.$$

EXAMPLES

1. In the curve of Ex. 1, p. 24, show that $P_0OM_3P_3 = \frac{15}{4}$.

Also $$P_1M_1M_2P_2 = \frac{11}{12}.$$

2. Find the area included between the equilateral hyperbola $2xy = a^2$, the axis of X, and two ordinates $x = a$, $x = 2\,a$.

Ans. $a^2 \log \sqrt{2}$.

3. Find the area included between the witch of Agnesi (Art. 126), the axes of X and Y, and the ordinate $x = 2\,a$. *Ans.* πa^2.

4. Find the area included between the catenary (Art. 128), the axis of X, and the ordinates $x = a$, $x = 2\,a$.

Ans. $\frac{a^2}{2}(e^2 - e + e^{-1} - e^{-2})$.

5. Find the area of one arch of $y = \sin x$. *Ans.* 2.

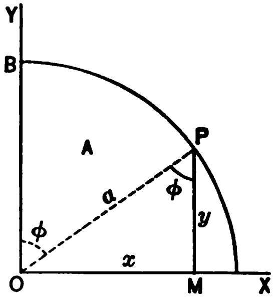
6. Find the area included between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ (Art. 129), and the axes of X and Y. *Ans.* $\frac{a^2}{6}$.

7. Find the area included between the semicubical parabola $ay^2 = x^3$ (Art. 130), the axis of Y, and two abscissas, $y = 8\,a$, $y = 27\,a$.

Ans. $\frac{633}{5}a^2$.

Conversely, instead of determining the area from the integral, we may find the integral from the area, when it can be obtained geometrically from the figure. For example:

8. Find $\int\sqrt{a^2-x^2}\,dx$, by means of the curve $y=\sqrt{a^2-x^2}$, circle about O, radius a.



$$A = BOMP = OMP + BOP$$

$$= \frac{1}{2}xy + \frac{a^2}{2}\phi = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

If the initial ordinate, instead of OB, had been some other ordinate, we should have had

$$A = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C, \text{ where } C \text{ is independent of } x.$$

Hence $A = \int y\,dx = \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

9. Find $\int(3x+2)dx$, by means of the line $y=3x+2$.

10. Find $\int\sqrt{2ax-x^2}\,dx$, by means of the curve $y=\sqrt{2ax-x^2}$.

Ans. $\int\sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + C.$

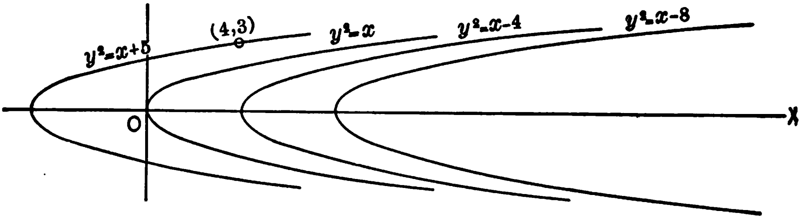
193. Other Illustrations. In order to further illustrate the determination of the constant of integration, we will work three examples, involving geometrical or physical properties.

Ex. 1. Determine the equation of a curve through the point (4, 3), at every point of which the slope of the tangent is equal to the reciprocal of twice the ordinate of the point of contact.

By the hypothesis $$\frac{dy}{dx}=\frac{1}{2y},$$

from which $$2ydy=dx.$$

Integrating, $$y^2=x+C. \quad \dots\dots\dots \quad (1)$$



This equation represents a series of parabolas whose axes coincide with the axis of x.

If now we impose the additional condition that the curve must pass through the point (4, 3), its coördinates must satisfy equation (1), giving
$$9=4+C, \quad C=5.$$

The equation of the particular curve is therefore
$$y^2=x+5.$$

Ex. 2. A body starting with a given initial velocity v_0, moves with a constant acceleration g. Find the space passed over in any time.

In Art. 19, acceleration $=\frac{dv}{dt}$.

Here $$\frac{dv}{dt}=g, \ dv=g\,dt.$$

Integrating, $$v=gt+C.$$

From the conditions of the example, $v=v_0$ when $t=0$; therefore
$$v_0=0+C, \quad C=v_0.$$

Hence $$v = gt + v_0.$$

Since $$v = \frac{ds}{dt} \text{ (Art. 18)}, \quad ds = gt\,dt + v_0\,dt.$$

Integrating, $$s = \frac{1}{2}gt^2 + v_0t + C.$$

From the conditions of the example, $s = 0$ when $t = 0$; therefore $C = 0$, and $s = \frac{1}{2}gt^2 + v_0t$ is the complete solution.

Ex. 3. A body is projected at an angle α with the horizon, and with a velocity v_0. Find the equation of its path.

Represent the horizontal and vertical components of the velocity by v_x and v_y respectively. Then, since gravity is the only force acting on the body, we have, g denoting the acceleration of gravity,

$$\frac{dv_x}{dt} = 0, \quad \text{and} \quad \frac{dv_y}{dt} = -g.$$

Integrating, $v_x = C$, $\quad v_y = -gt + C'$.

When $t = 0$, $v_x = v_0 \cos \alpha$, $\quad v_y = v_0 \sin \alpha$.

Hence $v_x = v_0 \cos \alpha$, $\quad v_y = -gt + v_0 \sin \alpha$;

that is, $\frac{dx}{dt} = v_0 \cos \alpha$, $\quad \frac{dy}{dt} = -gt + v_0 \sin \alpha$.

Integrating, $x = v_0t \cos \alpha + C$, $\quad y = -\frac{1}{2}gt^2 + v_0t \sin \alpha + C'$.

When $t = 0$, x and y, and therefore C and C', are zero.

Hence $x = v_0t \cos \alpha$, and $y = -\frac{1}{2}gt^2 + v_0t \sin \alpha$.

Eliminating t between these equations, we have as the equation of the path of the projectile,

$$y = x \tan \alpha - \frac{gx^2}{2\,v_0^2 \cos^2 \alpha}.$$

This evidently represents a parabola whose axis is parallel to the axis of Y.

EXAMPLES

4. Find the equation of the curve whose subnormal (Art. 146) has the constant value 4, and which passes through the point (1, 4).
Ans. $y^2 = 8x + 8$.

5. Find the equation of the curve whose subtangent (Art. 146) is twice the abscissa of the point of contact, and which passes through the point (2, 1).
Ans. $x = 2y^2$.

6. The slope of the tangent to a curve at any point is $-\frac{4x}{9y}$, and the curve passes through the point (3, 2). Find its equation.
Ans. $4x^2 + 9y^2 = 72$.

7. Find the equation of the curve whose polar subtangent (Art. 153) is 3 times the length of the corresponding radius vector, and which passes through the point (2, 0).
Ans. $r = 2e^{\frac{\theta}{3}}$.

8. Find the equation of the curve whose polar subnormal (Art. 153) is 3 times the length of the corresponding radius vector, and which passes through the point (2, 0).
Ans. $r = 2e^{3\theta}$.

9. Find the equation of a curve through the point $\left(3, \frac{\pi}{3}\right)$, in which the angle between the radius vector and the tangent is half the vectorial angle.
Ans. $r = 6(1 - \cos\theta)$.

10. A balloon is ascending with a velocity of 20 miles an hour. A stone dropped from the balloon reaches the ground in 6 seconds. Find the height of the balloon when the stone is dropped, calling the acceleration of gravity 32 ft. per sec.
Ans. 400 ft.

11. If a particle moves so that its velocities parallel to the axes of X and Y are ky and kx respectively, prove that its path is an equilateral hyperbola.

12. A body starts from the origin of coördinates, and in t seconds its velocity parallel to the axis of X is $6t$, and its velocity parallel to the axis of Y is $3t^2-3$. Find (*a*) the distances traversed parallel to each axis in t seconds; (*b*) the distance traversed along the path (Ex. 41, p. 38); (*c*) the equation of the path.

Ans. (*a*) $x=3t^2$; $y=t^3-3t$.

(*b*) $s=t^3+3t$.

(*c*) $27y^2=x(x-9)^2$.

13. A body is projected directly upward with an initial velocity of 400 feet per second. Neglecting the resistance of the air, find how high it will rise. *Ans.* 2500 ft.

Find its height when its velocity is reduced one-half.

Ans. 1875 ft.

14. If a body projected from the top of a tower at an angle of 45° above the horizontal plane, falls in 5 seconds at a distance from the bottom of the tower equal to its height, find the height of the tower ($g=32$). *Ans.* 200 ft.

CHAPTER XXII

INTEGRATION OF RATIONAL FRACTIONS

194. Formulæ for Integration of Rational Functions. On examining the fundamental integrals in Art. 186, it will be seen that only four apply to the integration of rational algebraic functions, I., II., XV., and XVI.; and of these only I., II., and XV. are independent, since XVI. depends directly upon II.

It will be shown in this chapter that by these three formulæ any rational function can be integrated. The integration of a rational polynomial has been explained in Chapter XX. We will now consider the integration of rational fractions.

195. Preliminary Operation. If the degree of the numerator is equal to, or greater than, that of the denominator, the fraction should be reduced to a mixed quantity, by dividing the numerator by the denominator.

For example,

$$\frac{x^3-2x^2}{x^3+1}=1-\frac{2x^2+1}{x^3+1}.$$

$$\frac{2x^5-3x^4+1}{x^4+x^2}=2x-3+\frac{-2x^3+3x^2+1}{x^4+x^2}.$$

The degree of the numerator of the new fraction will be less than that of the denominator.

The entire part of the mixed quantity is readily integrable, and thus the integration of any rational fraction is made to depend upon the integration of one whose numerator is of a lower degree than the denominator.

196. Partial Fractions. A rational fraction is integrated by decomposing it into partial fractions, whose denominators are the factors of the original denominator. The complete discussion of Partial Fractions belongs to Algebra. We shall only consider here the form of these partial fractions and the processes of determining them.

Factors of the Denominator. It is shown by the Theory of Equations that a polynomial of the nth degree, with respect to x, may be resolved into n factors of the first degree,

$$(x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n).$$

These factors are real or imaginary, but the imaginary factors occur in pairs, of the form

$$x-a+b\sqrt{-1}, \text{ and } x-a-b\sqrt{-1},$$

whose product is $(x-a)^2+b^2$, a real factor of the second degree.

It follows that any polynomial may be resolved into real factors of the first or second degree, and only such factors will be considered in the denominators of fractions.

There are four cases to be considered.

First. Where the denominator contains factors of the *first* degree only, each of which occurs but once.

Second. Where the denominator contains factors of the *first* degree only, some of which are repeated.

Third. Where the denominator contains factors of the *second* degree, each of which occurs but once.

Fourth. Where the denominator contains factors of the *second* degree, some of which are repeated.

197. CASE I. **Factors of the Denominator all of the First Degree, and none repeated.**

The given fraction may be decomposed into partial fractions, as shown by the following example,

$$\int\frac{x^2+6x-8}{x^3-4x}dx.$$

Assume

$$\frac{x^2+6x-8}{x^3-4x}=\frac{x^2+6x-8}{(x-2)(x+2)x}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x}, \quad . \quad . \quad (1)$$

where A, B, C are unknown constants.

Clearing (1) of fractions,

$$x^2+6x-8=Ax(x+2)+Bx(x-2)+C(x-2)(x+2) \quad . \quad . \quad . \quad (2)$$

$$=(A+B+C)x^2+2(A-B)x-4C.$$

Equating the coefficients of like powers of x in the two members of the equation, according to the method of Undetermined Coefficients, we have

$$A+B+C=1, \quad 2(A-B)=6, \quad -4C=-8.$$

whence $\qquad A=1, \quad B=-2, \quad C=2.$

Hence $$\frac{x^2+6x-8}{x^3-4x}=\frac{1}{x-2}-\frac{2}{x+2}+\frac{2}{x},$$

and $$\int\frac{x^2+6x-8}{x^3-4x}dx=\log(x-2)-2\log(x+2)+2\log x$$

$$=\log\frac{x^2(x-2)}{(x+2)^2}.$$

The following is a shorter method of finding A, B, C:

Suppose the denominator of the given fraction to contain the factor $x-a$, not repeated. Then the fraction may be expressed as

$$\frac{f(x)}{(x-a)\phi(x)}=\frac{A}{x-a}+\frac{\psi(x)}{\phi(x)}.$$

Hence $$\frac{f(x)}{\phi(x)}=A+(x-a)\frac{\psi(x)}{\phi(x)}.$$

This being an identical equation is true for all values of x.

If we put $x=a$, we have $A=\frac{f(a)}{\phi(a)}$, since by hypothesis $\phi(x)$ does not vanish when $x=a$.

Thus we have the following rule:

To find A, the numerator of the partial fraction $\frac{A}{x-a}$, put $x=a$ in the given fraction, omitting the factor $x-a$ itself.

For example, having written equation (1), we find A by substituting $x=2$ in the given fraction $\frac{x^2+6x-8}{(x-2)(x+2)x}$, omitting the factor $x-2$. This gives

$$A=\frac{4+12-8}{4(2)}=1.$$

To find B, substitute $x=-2$, omitting the factor $x+2$.

$$B=\frac{4-12-8}{-4(-2)}=-2.$$

To find C, substitute $x=0$, omitting the factor x.

$$C=\frac{-8}{-4}=2.$$

EXAMPLES

The constant of integration C will be omitted in the examples in this chapter, and the following chapters on the integration of functions.

1. $\int\frac{x^4\,dx}{x^2-3x+2}=\frac{x^3}{3}+\frac{3x^2}{2}+7x+\log\frac{(x-2)^{16}}{x-1}.$

2. $\int\frac{(x^2+x+1)dx}{x^3-4x^2+x+6}=\frac{1}{12}\log\frac{(x+1)(x-3)^{39}}{(x-2)^{28}}.$

3. $\int\frac{(w+1)^2dw}{4w^3-w}=\frac{1}{8}\log\frac{(2w+1)(2w-1)^9}{w^8}.$

4. $$\int\frac{x\,dx}{(x+1)(x+3)(x+5)}=\frac{1}{8}\log\frac{(x+3)^6}{(x+1)(x+5)^5}.$$

5. $$\int\frac{x^2\,dx}{(2x-1)(3x-1)(3x-2)}=\frac{1}{18}\log\frac{(3x-1)^2(3x-2)^8}{(2x-1)^9}.$$

6. $$\int\frac{ax+bx}{(ax-b)(bx-a)}dx=\frac{a}{ab-b^2}\log(bx-a)+\frac{b}{ab-a^2}\log(ax-b).$$

7. $$\int\frac{(x+a)^2dx}{2x^3-ax^2-a^2x}=\frac{1}{6}\log\frac{(2x+a)(x-a)^8}{x^6}.$$

8. $$\int\frac{(x+a+b)^2dx}{(x+a)(x+b)}=x+\frac{b^2}{b-a}\log(x+a)+\frac{a^2}{a-b}\log(x+b).$$

9. $$\int\frac{dy}{(y+n)(y^2-1)}=\frac{1}{2(n^2-1)}\log\frac{(y+n)^2(y-1)^{n-1}}{(y+1)^{n+1}}.$$

10. $$\int\frac{(a^2-b^2)(x^2+1)dx}{(a^2x^2-b^2)(b^2x^2-a^2)}=\frac{1}{2ab}\log\frac{(ax+b)(bx-a)}{(ax-b)(bx+a)}.$$

11. $$\int\frac{(x+1)dx}{(x^2-19)^2-4(x+8)^2}=\frac{1}{360}\log\frac{(x+5)^{10}(x-7)^4}{(x+3)^9(x-1)^5}.$$

12. $$\int\frac{(x+1)dx}{4x^5-17x^3+4x}$$

$$=\frac{1}{120}\log\frac{(x-2)^3}{x+2}-\frac{1}{15}\log\left[(2x+1)(2x-1)^3\right]+\frac{1}{4}\log x.$$

198. CASE II. **Factors of the denominator all of the first degree, and some repeated.**

Here the method of decomposition of Case I. requires modification. Suppose, for example, we have

$$\int \frac{x^3+1}{x(x-1)^3}dx.$$

If we follow the method of the preceding case, we should write

$$\frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-1}+\frac{D}{x-1}.$$

But since the common denominator of the fractions in the second member of this equation is $x(x-1)$, their sum cannot be equal to the given fraction with the denominator $x(x-1)^3$. To meet this objection, we assume

$$\frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{(x-1)^3}+\frac{C}{(x-1)^2}+\frac{D}{x-1}.$$

Clearing of fractions,

$$x^3+1=A(x-1)^3+Bx+Cx(x-1)+Dx(x-1)^2$$

$$=(A+D)x^3+(-3A+C-2D)x^2+(3A+B-C+D)x-A$$

Hence $$A+D=1, \quad . \quad . \quad . \quad (1)$$

$$-3A+C-2D=0, \quad . \quad . \quad . \quad (2)$$

$$3A+B-C+D=0,$$

$$-A=1.$$

Whence $A=-1,\ B=2,\ C=1,\ D=2.$

Therefore $$\frac{(x^3+1)}{x(x-1)^3}=-\frac{1}{x}+\frac{2}{(x-1)^3}+\frac{1}{(x-1)^2}+\frac{2}{x-1}.$$

Hence
$$\int \frac{x^3+1}{x(x-1)^3}dx = -\log x - \frac{1}{(x-1)^2} - \frac{1}{x-1} + 2\log(x-1)$$
$$= -\frac{x}{(x-1)^2} + \log\frac{(x-1)^2}{x}.$$

The numerators A and B may be determined by the short method given for Case I., and then C and D may be found by (1) and (2).

EXAMPLES

1. $\displaystyle\int \frac{x^5 - x^3 + 1}{x^4 - x^3}dx = \frac{x^2}{2} + x + \frac{1}{2x^2} + \frac{1}{x} + \log\frac{x-1}{x}.$

2. $\displaystyle\int \frac{x^2dx}{(x+1)(x-1)^3} = \frac{2-3x}{4(x-1)^2} + \frac{1}{8}\log\frac{x-1}{x+1}.$

3. $\displaystyle\int \frac{8\,dx}{x(x^2-4)^2} = -\frac{1}{x^2-4} + \frac{1}{4}\log\frac{x^2}{x^2-4}.$

4. $\displaystyle\int \frac{(19x-32)dx}{(4x+1)(2x-3)^2} = \frac{1}{4(2x-3)} + \frac{3}{4}\log\frac{2x-3}{4x+1}.$

5. $\displaystyle\int \frac{x^2dx}{(9x^2-4)^2} = -\frac{x}{18(9x^2-4)} + \frac{1}{216}\log\frac{3x-2}{3x+2}.$

6. $\displaystyle\int \frac{x^3+a^3}{(x^2-ax)^2}dx = \frac{a^2-3ax}{x^2-ax} + \log\frac{x^2}{x-a}.$

7. $\displaystyle\int \left(\frac{x+1}{x-1}\right)^4dx = x + \frac{16(2-3x)}{3(x-1)^3} - \frac{24}{x-1} + 8\log(x-1)$

8. $$\int\left(\frac{x+a}{x+b}\right)^3 dx = x - \frac{(a-b)^3}{2(x+b)^2} - \frac{3(a-b)^2}{x+b} + 3(a-b)\log(x+b).$$

9. $$\int\frac{x\,dx}{(x^2-4x+1)^2} = -\frac{2x-1}{6(x^2-4x+1)} + \frac{1}{6\sqrt{3}}\log\frac{x-2+\sqrt{3}}{x-2-\sqrt{3}}.$$

10. $$\int\frac{(x+a+b)^3}{(x+a)^2(x+b)^2}dx = -\frac{1}{(a-b)^2}\left(\frac{b^3}{x+a}+\frac{a^3}{x+b}\right)$$
$$+\frac{3ab^2-b^3}{(a-b)^3}\log(x+a)+\frac{a^3-3a^2b}{(a-b)^3}\log(x+b).$$

199. CASE III. **Denominator containing Factors of the Second Degree, but none repeated.**

The form of decomposition will appear from the following example,

$$\int\frac{5x+12}{x(x^2+4)}dx.$$

We assume $$\frac{5x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}, \quad \dots\dots\dots \quad (1)$$

and in general for every partial fraction in this case, whose denominator is of the second degree, we must assume a numerator of the form $Bx+C$.

Clearing (1) of fractions,

$$5x+12 = (A+B)x^2 + Cx + 4A.$$

$$A+B=0,\ C=5,\ 4A=12.$$

Whence $A=3,\ B=-3,\ C=5;$

therefore $$\frac{5x+12}{x(x^2+4)} = \frac{3}{x} + \frac{-3x+5}{x^2+4}.$$

$$\int\frac{-3x+5}{x^2+4}dx = -3\int\frac{x\,dx}{x^2+4} + 5\int\frac{dx}{x^2+4}$$

$$= -\frac{3}{2}\log(x^2+4) + \frac{5}{2}\tan^{-1}\frac{x}{2}.$$

Hence $$\int \frac{5x+12}{x(x^2+4)}dx = 3\log\frac{x}{\sqrt{x^2+4}}+\frac{5}{2}\tan^{-1}\frac{x}{2}.$$

Take for another example,

$$\int \frac{(2x^2-3x-3)dx}{(x-1)(x^2-2x+5)}.$$

This fraction is decomposed as follows:

$$\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)} = -\frac{1}{x-1}+\frac{3x-2}{x^2-2x+5}.$$

$$\int\frac{(3x-2)dx}{x^2-2x+5} = \int\frac{(3x-3)dx}{x^2-2x+5}+\int\frac{dx}{x^2-2x+5}$$

$$= \frac{3}{2}\log(x^2-2x+5)+\frac{1}{2}\tan^{-1}\frac{x-1}{2}.$$

$$\int\frac{(2x^2-3x-3)dx}{(x-1)(x^2-2x+5)} = \log\frac{(x^2-2x+5)^{\frac{3}{2}}}{x-1}+\frac{1}{2}\tan^{-1}\frac{x-1}{2}.$$

The integration of any fraction with a quadratic denominator like the preceding, $\int\frac{(3x-2)dx}{x^2-2x+5}$, may be shown as follows:

Having written the denominator in the form $(x+a)^2+b^2$, we have

$$\int\frac{(px+q)dx}{(x+a)^2+b^2} = \int\frac{p(x+a)dx}{(x+a)^2+b^2}+\int\frac{(q-pa)dx}{(x+a)^2+b^2}$$

$$= \frac{p}{2}\log\left[(x+a)^2+b^2\right]+\frac{q-pa}{b}\tan^{-1}\frac{x+a}{b}.$$

EXAMPLES

1. $$\int\frac{32x^5+3}{4x^3+3x}dx=\frac{8x^3}{3}-6x+\frac{1}{2}\log\frac{x^2}{4x^2+3}+3\sqrt{3}\tan^{-1}\frac{2x}{\sqrt{3}}.$$

2. $$\int\frac{(2x^2-x)dx}{x^4+x^2-2}=\frac{1}{6}\log\frac{(x-1)(x^2+2)}{(x+1)^3}+\frac{2\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}.$$

3. $$\int\frac{4x^2-3x+1}{2x^5+x^3}dx=\frac{6x-1}{2x^2}+\log\frac{x^2}{2x^2+1}+3\sqrt{2}\tan^{-1}(x\sqrt{2}).$$

4. $$\int\frac{x^6dx}{x^4-1}=\frac{x^3}{3}+\frac{1}{4}\log\frac{x-1}{x+1}+\frac{1}{2}\tan^{-1}x.$$

When the given fraction and the denominators of the partial fractions contain only *even* powers of x, they may be regarded as functions of x^2, and we may assume A, B, C, etc., as the numerators of the partial fractions.

In the following example, the partial fractions may be assumed as

$$\frac{A}{x^2+a^2}+\frac{B}{x^2+b^2}.$$

5. $$\int\frac{x^4dx}{(x^2+a^2)(x^2+b^2)}=x+\frac{1}{a^2-b^2}\left(b^3\tan^{-1}\frac{x}{b}-a^3\tan^{-1}\frac{x}{a}\right).$$

6. $$\int\frac{(1-x^2)dx}{(4x^2+1)(x^2+4)}=\frac{1}{6}\left(\tan^{-1}2x-\tan^{-1}\frac{x}{2}\right)=\frac{1}{6}\tan^{-1}\frac{3x}{2+2x^2}.$$

7. $$\int\frac{(a^2+b^2)(1+x^2)}{(a^2x^2+b^2)(b^2x^2+a^2)}dx=\frac{1}{ab}\tan^{-1}\frac{(a^2+b^2)x}{ab(1-x^2)}.$$

8. $$\int\frac{(x-1)dx}{x(x^2-6x+13)}=\frac{1}{26}\log\frac{x^2-6x+13}{x^2}+\frac{5}{13}\tan^{-1}\frac{x-3}{2}.$$

9. $$\int \frac{(3x^3-2x-20)dx}{(x^2+3)(2x^2-6x+5)} = \frac{1}{4}\log\frac{(2x^2-6x+5)^5}{(x^2+3)^2} + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{5}{2}\tan^{-1}(2x-3).$$

10. $$\int \frac{y\,dy}{y^3-1} = \frac{1}{6}\log\frac{(y-1)^2}{y^2+y+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}.$$

11. $$\int \frac{(x^2+2)dx}{x^4+x^2+1} = \frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1} + \frac{\sqrt{3}}{2}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}.$$

12. $$\int \frac{w^2\,dw}{w^4+1} = \frac{1}{4\sqrt{2}}\log\frac{w^2-w\sqrt{2}+1}{w^2+w\sqrt{2}+1} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{w\sqrt{2}}{1-w^2}.$$

13. $$\int \frac{dx}{x^5-x^3+x^2-1} = \frac{1}{6(x+1)} + \frac{1}{4}\log\frac{x-1}{x+1} - \frac{2}{3\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

200. Case IV. **Denominator containing Factors of the Second Degree, some of which are repeated.**

This case is related to Case III., as Case II. to Case I., and requires a similar modification of the partial fractions.

For illustration take

$$\int \frac{2x^3+x^2+3}{(x^2+1)^2}\,dx.$$

We assume

$$\frac{2x^3+x^2+3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}.$$

$$2x^3+x^2+3 = Cx^3 + Dx^2 + (A+C)x + B + D.$$

$$A=-2,\ B=2,\ C=2,\ D=1.$$

Therefore $$\frac{2x^3+x^2+3}{(x^2+1)^2}=\frac{-2x+2}{(x^2+1)^2}+\frac{2x+1}{x^2+1}$$

$$\int\frac{-2x+2}{(x^2+1)^2}dx=-\int\frac{2x\,dx}{(x^2+1)^2}+2\int\frac{dx}{(x^2+1)^2}$$

$$=\frac{1}{x^2+1}+2\int\frac{dx}{(x^2+1)^2}.$$

To integrate the last fraction, we use the following formula of reduction,

$$\int\frac{dx}{(x^2+a^2)^n}=\frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}}+(2n-3)\int\frac{dx}{(x^2+a^2)^{n-1}}\right].*$$

This formula enables us to integrate $\int\frac{dx}{(x^2+a^2)^n}$ by making it depend upon $\int\frac{dx}{(x^2+a^2)^{n-1}}$. By successive applications the given integral is made to depend ultimately upon $\int\frac{dx}{x^2+a^2}$, which is $\frac{1}{a}\tan^{-1}\frac{x}{a}$.

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* This formula may be derived as follows:

$$\frac{d}{dx}\left[\frac{x}{(x^2+a^2)^n}\right]=\frac{d}{dx}[x(x^2+a^2)^{-n}]=(x^2+a^2)^{-n}-2nx^2(x^2+a^2)^{-n-1}$$
$$=(x^2+a^2)^{-n}-2n[(x^2+a^2)-a^2](x^2+a^2)^{-n-1}$$
$$=(1-2n)(x^2+a^2)^{-n}+2na^2(x^2+a^2)^{-n-1}.$$

Integrating both members after multiplying by dx,

$$\frac{x}{(x^2+a^2)^n}=(1-2n)\int\frac{dx}{(x^2+a^2)^n}+2na^2\int\frac{dx}{(x^2+a^2)^{n+1}},$$

$$2na^2\int\frac{dx}{(x^2+a^2)^{n+1}}=\frac{x}{(x^2+a^2)^n}+(2n-1)\int\frac{dx}{(x^2+a^2)^n}.$$

Substituting for n, $n-1$, we have

$$2(n-1)a^2\int\frac{dx}{(x^2+a^2)^n}=\frac{x}{(x^2+a^2)^{n-1}}+(2n-3)\int\frac{dx}{(x^2+a^2)^{n-1}}.$$

Substituting in the formula $n=2$ and $a^2=1$, we have

$$\int\frac{dx}{(x^2+1)^2}=\frac{1}{2}\left[\frac{x}{x^2+1}+\int\frac{dx}{x^2+1}\right]=\frac{x}{2(x^2+1)}+\frac{1}{2}\tan^{-1}x;$$

whence
$$\int\frac{-2x+2}{(x^2+1)^2}\,dx=\frac{1}{x^2+1}+\frac{x}{x^2+1}+\tan^{-1}x,$$

and
$$\int\frac{2x^3+x^2+3}{(x^2+1)^2}\,dx=\frac{x+1}{x^2+1}+2\tan^{-1}x+\log(x^2+1).$$

A partial fraction of the form $\frac{px+q}{[(x+a)^2+b^2]^n}$, by substituting $x+a=z$, becomes $\frac{pz+q-pa}{(z^2+b^2)^n}$, the integration of which has already been explained.

For example, if $x-3=z$, $\int\frac{(5x+1)\,dx}{(x^2-6x+12)^3}=\int\frac{5z+16}{(z^2+3)^3}\,dz,$

$$=-\frac{5}{4(z^2+3)^2}+16\int\frac{dz}{(z^2+3)^3}.$$

By the formula of reduction,

$$\int\frac{dz}{(z^2+3)^3}=\frac{1}{12}\left[\frac{z}{(z^2+3)^2}+3\int\frac{dz}{(z^2+3)^2}\right]$$

$$=\frac{z}{12(z^2+3)^2}+\frac{1}{4}\cdot\frac{1}{6}\left[\frac{z}{z^2+3}+\int\frac{dz}{z^2+3}\right]$$

$$=\frac{z}{12(z^2+3)^2}+\frac{z}{24(z^2+3)}+\frac{1}{24\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}.$$

$$\int\frac{5z+16}{(z^2+3)^3}\,dz=\frac{16z-15}{12(z^2+3)^2}+\frac{2z}{3(z^2+3)}+\frac{2}{3\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}.$$

Hence
$$\int\frac{(5x+1)\,dx}{(x^2-6x+12)^3}=\frac{16x-63}{12(x^2-6x+12)^2}$$

$$+\frac{2(x-3)}{3(x^2-6x+12)}+\frac{2}{3\sqrt{3}}\tan^{-1}\frac{x-3}{\sqrt{3}}.$$

EXAMPLES

1. $$\int\frac{x^5+x^3-1}{x^2(x^2+1)^2}dx=\frac{3x^2+2}{2x(x^2+1)}+\frac{1}{2}\log(x^2+1)+\frac{3}{2}\tan^{-1}x.$$

2. $$\int\left(\frac{x^2-a^2}{x^2+a^2}\right)^2dx=x+\frac{2a^2x}{x^2+a^2}-2a\tan^{-1}\frac{x}{a}.$$

3. $$\int\frac{4x+3}{(4x^2+3)^3}dx=\frac{4x^3+5x-2}{8(4x^2+3)^2}+\frac{1}{16\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}.$$

For the following example, see note preceding Ex. 5, Case III.

4. $$\int\frac{36x^2(x^2+1)^2+25x^2}{(4x^2+9)^2(9x^2+4)^2}dx=-\frac{x^3+x}{2(4x^2+9)(9x^2+4)}$$
$$+\frac{1}{156}\tan^{-1}\frac{13x}{6-6x^2}.$$

5. $$\int\frac{x^3+8x-21}{(x^2+4x+9)^2}dx=-\frac{3(x+7)}{2(x^2+4x+9)}$$
$$+\frac{1}{2}\log(x^2+4x+9)-\frac{3\sqrt{5}}{2}\tan^{-1}\frac{x+2}{\sqrt{5}}.$$

6. $$\int\frac{(x^3-2)dx}{(x^2+x+1)(x^2+x+2)^2}=-\frac{x+4}{7(x^2+x+2)}$$
$$+\frac{12}{7\sqrt{7}}\tan^{-1}\frac{2x+1}{\sqrt{7}}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

7. $$\int\frac{9x^3dx}{(x^3+1)^2}=-\frac{3x}{x^3+1}+\frac{1}{2}\log\frac{(x+1)^2}{x^2-x+1}+\sqrt{3}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

8. $$\int\frac{dx}{[(x+2)^4-(x+1)^4]^2}=-\frac{12x^2+36x+29}{2(2x+3)(2x^2+6x+5)}$$
$$-3\tan^{-1}(2x+3).$$

CHAPTER XXIII

INTEGRATION OF IRRATIONAL FUNCTIONS

201. We have shown in the preceding chapter that the integral of any rational function can be expressed in terms of algebraic, logarithmic, and inverse-trigonometric functions.

We shall now consider the integration of irrational functions.

202. Integration by Rationalization. Some integrals involving radicals may be integrated, by reducing them to rational integrals by a change of variable. This is possible, however, in only a very limited number of cases. This process is sometimes called *integration by rationalization.*

203. Integrals containing $(ax+b)^{\frac{p}{q}}$. Such an integral may be rationalized by the substitution $ax+b=z^q$.

For example, take
$$\int\frac{x^2dx}{(2x+3)^{\frac{2}{3}}}.$$

Assume
$$2x+3=z^3,\quad x=\frac{z^3-3}{2},\quad dx=\frac{3z^2dz}{2}.$$

Then
$$\int\frac{x^2dx}{(2x+3)^{\frac{2}{3}}}=\int\frac{\left(\frac{z^3-3}{2}\right)^2\frac{3z^2dz}{2}}{z^2}=\frac{3}{8}\int(z^3-3)^2dz$$

$$=\frac{3}{8}\left(\frac{z^7}{7}-\frac{3z^4}{2}+9z\right)=\frac{3z}{8}\left(\frac{z^6}{7}-\frac{3z^3}{2}+9\right)=\frac{3}{112}(2x+3)^{\frac{1}{3}}(8x^2-18x+81).$$

Another example is $$\int\frac{x\,dx}{\sqrt{x}+1}.$$

Assume $$x=z^2,\quad dx=2\,z\,dz.$$

Then $$\int\frac{x\,dx}{\sqrt{x}+1}=\int\frac{2\,z^3\,dz}{z+1}=2\int\left(z^2-z+1-\frac{1}{z+1}\right)dz$$

$$=2\left[\frac{z^3}{3}-\frac{z^2}{2}+z-\log(z+1)\right]=\frac{2}{3}x^{\frac{3}{2}}-x+2\,x^{\frac{1}{2}}-\log(x^{\frac{1}{2}}+1)^2$$

204. Integrals containing $(ax+b)^{\frac{p}{q}}$, $(ax+b)^{\frac{r}{s}}$, $\cdots$. In this case the integral is rationalized by the substitution $ax+b=z^n$, where n is the least common multiple of q, s, $\cdots$, the denominators of the fractional exponents.

Take, for example, $$\int\frac{dx}{(x-2)^{\frac{1}{2}}+(x-2)^{\frac{2}{3}}}.$$

Assume $$x-2=z^6,\quad dx=6\,z^5\,dz,$$

$$(x-2)^{\frac{1}{2}}=z^3,\quad (x-2)^{\frac{2}{3}}=z^4.$$

$$\int\frac{dx}{(x-2)^{\frac{1}{2}}+(x-2)^{\frac{2}{3}}}=\int\frac{6\,z^5\,dz}{z^3+z^4}=6\int\frac{z^2\,dz}{z+1}=6\int\left(z-1+\frac{1}{z+1}\right)dz$$

$$=6\left[\frac{z^2}{2}-z+\log(z+1)\right]=3(x-2)^{\frac{1}{3}}-6(x-2)^{\frac{1}{6}}+6\log\left[(x-2)^{\frac{1}{6}}+1\right]$$

EXAMPLES

1. $\displaystyle\int\frac{x+1}{x\sqrt{x-2}}dx=2\sqrt{x-2}+\sqrt{2}\tan^{-1}\sqrt{\frac{x-2}{2}}.$

2. $\displaystyle\int x^2\sqrt[4]{4\,x-3}\,dx=\frac{(4\,x-3)^{\frac{5}{4}}}{8}\left[\frac{(4\,x-3)^2}{26}+\frac{4\,x-3}{3}+\frac{9}{10}\right].$

3. $\displaystyle\int\frac{dx}{1+\sqrt[3]{3\,x-2}}=\frac{1}{2}(3\,x-2)^{\frac{2}{3}}-(3\,x-2)^{\frac{1}{3}}+\log(1+\sqrt[3]{3x-2}).$

4. $$\int\frac{x^{\frac{1}{4}}+x^{\frac{1}{6}}+1}{x^{\frac{2}{3}}+x^{\frac{5}{12}}}\,dx=\frac{12\,x^{\frac{7}{12}}}{7}+2\,x^{\frac{1}{2}}-4\,x^{\frac{1}{4}}+4\log\left(x^{\frac{1}{4}}+1\right).$$

5. $$\int\frac{y^3\,dy}{(4\,y+1)^{\frac{5}{2}}}=\frac{4\,y^3-6\,y^2-6\,y-1}{24(4\,y+1)^{\frac{3}{2}}}.$$

6 $$\int\frac{dw}{w+\sqrt{2\,w-1}}=\frac{2}{\sqrt{2\,w-1}+1}+2\log\left(\sqrt{2\,w-1}+1\right).$$

7. $$\int\frac{dt}{t^{\frac{2}{3}}+2\,t^{\frac{3}{4}}}=2\,t^{\frac{1}{4}}-\frac{3\,t^{\frac{1}{6}}}{2}+\frac{3\,t^{\frac{1}{12}}}{2}-\frac{3}{4}\log\left(2\,t^{\frac{1}{12}}+1\right).$$

8. $$\int x^2\sqrt{ax+b}\,dx=\frac{2(ax+b)^{\frac{3}{2}}}{105\,a^3}\left(15\,a^2x^2-12\,abx+8\,b^2\right).$$

9. $$\int\frac{dx}{(3\,x+1)^{\frac{5}{6}}+4(3\,x+1)^{\frac{1}{2}}}=2(3\,x+1)^{\frac{1}{6}}-4\tan^{-1}\frac{(3\,x+1)^{\frac{1}{6}}}{2}.$$

10. $$\int\frac{\sqrt{x+1}-1}{x+3}\,dx=2\sqrt{x+1}-\log\left(x+3\right)-2\sqrt{2}\tan^{-1}\sqrt{\frac{x+1}{2}}.$$

11. $$\int\frac{(x-2)dx}{(2\,x-3)^{\frac{5}{3}}+6\,x-9}=\frac{3}{4}(2\,x-3)^{\frac{1}{3}}+\frac{1}{8}\log\frac{(2\,x-3)^{\frac{2}{3}}+3}{(2\,x-3)^{\frac{2}{3}}}$$
$$-\frac{3\sqrt{3}}{4}\tan^{-1}\frac{(2\,x-3)^{\frac{1}{3}}}{\sqrt{3}}.$$

12. $$\int\frac{dx}{\sqrt{2\,x+1}+\sqrt{x-1}}$$
$$=2\sqrt{2\,x+1}-2\sqrt{x-1}+2\sqrt{3}\left(\tan^{-1}\sqrt{\frac{x-1}{3}}-\tan^{-1}\sqrt{\frac{2\,x+1}{3}}\right)$$
$$=2\sqrt{2\,x+1}-2\sqrt{x-1}+\sqrt{3}\left(\cos^{-1}\frac{4-x}{x+2}-\cos^{-1}\frac{1-x}{x+2}\right).$$

13. $$\int\frac{dx}{(2x^{\frac{1}{4}}-1)(x^{\frac{1}{2}}+2)}=2x^{\frac{1}{4}}+\frac{1}{9}\log(2x^{\frac{1}{4}}-1)+\frac{4}{9}\log(x^{\frac{1}{2}}+2)$$
$$-\frac{16\sqrt{2}}{9}\tan^{-1}\frac{x^{\frac{1}{4}}}{\sqrt{2}}.$$

14. $$\int\frac{(x+2)dx}{(x+1)^{\frac{3}{2}}+1}=2\sqrt{x+1}+\frac{2}{3}\log(x+2-\sqrt{x+1})$$
$$-\frac{4}{3}\log(1+\sqrt{x+1}).$$

205. Roots of Polynomials of Higher Degrees.—In the rationalization of irrational integrals we now pass from roots of binomials of the first degree to roots of polynomials of higher degrees.

Here rationalization is limited to the *square* root of an expression of the *second* degree.

206. Integrals containing $\sqrt{x^2+ax+b}$. This may be rationalized by the substitution

$$\sqrt{x^2+ax+b}=z-x.$$

For example, consider $\displaystyle\int\frac{dx}{x\sqrt{x^2-x+2}}$.

If, following the method of the preceding articles, we assume

$$\sqrt{x^2-x+2}=z,\quad x^2-x+2=z^2,$$

the expression for x, and consequently that for dx, in terms of z, will involve radicals. This difficulty is avoided by assuming

$$\sqrt{x^2-x+2}=z-x,\quad -x+2=z^2-2zx,$$

cancelling x^2 in both members.

$$x=\frac{z^2-2}{2z-1},\qquad dx=\frac{2(z^2-z+2)dz}{(2z-1)^2},$$

$$\sqrt{x^2-x+2}=z-x=\frac{z^2-z+2}{2z-1}.$$

Hence,

$$\int\frac{dx}{x\sqrt{x^2-x+2}}=\int\frac{\dfrac{2(z^2-z+2)dz}{(2z-1)^2}}{\dfrac{z^2-2}{2z-1}\cdot\dfrac{z^2-z+2}{2z-1}}=\int\frac{2\,dz}{z^2-2}=\frac{1}{\sqrt{2}}\log\frac{z-\sqrt{2}}{z+\sqrt{2}}.$$

Substituting $z=\sqrt{x^2-x+2}+x,$

$$\int\frac{dx}{x\sqrt{x^2-x+2}}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{x^2-x+2}+x-\sqrt{2}}{\sqrt{x^2-x+2}+x+\sqrt{2}}.$$

207. Integrals containing $\sqrt{-x^2+ax+b}$. This may be rationalized by the substitution

$$\sqrt{-x^2+ax+b}=\sqrt{(\alpha-x)(\beta+x)}=(\alpha-x)z \text{ or } =(\beta+x)z,$$

where $\alpha-x$ and $\beta+x$ are the factors of $-x^2+ax+b$.

These factors will be real, unless $\sqrt{-x^2+ax+b}$ is imaginary for all values of x.

Take, for example, $\displaystyle\int\frac{dx}{x\sqrt{2+x-x^2}}$.

Assume $\sqrt{2+x-x^2}=\sqrt{(2-x)(1+x)}=(2-x)z.$

$$1+x=(2-x)z^2,\quad x=\frac{2z^2-1}{z^2+1},\quad dx=\frac{6z\,dz}{(z^2+1)^2}.$$

$$\sqrt{2+x-x^2}=(2-x)z=\frac{3z}{z^2+1}.$$

Therefore,

$$\int\frac{dx}{x\sqrt{2+x-x^2}}=\int\frac{2\,dz}{2z^2-1}=\frac{1}{\sqrt{2}}\log\frac{z\sqrt{2}-1}{z\sqrt{2}+1}.$$

Substituting $z=\sqrt{\dfrac{1+x}{2-x}},$

$$\int\frac{dx}{x\sqrt{2+x-x^2}}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2+2x}-\sqrt{2-x}}{\sqrt{2+2x}+\sqrt{2-x}}.$$

EXAMPLES

1. $$\int \frac{dx}{x\sqrt{x^2+4x-4}} = \tan^{-1}\frac{x+\sqrt{x^2+4x-4}}{2}.$$

2. $$\int \frac{\sqrt{x^2+4x}}{x^2}dx = -\frac{8}{x+\sqrt{x^2+4x}} + \log(x+2+\sqrt{x^2+4x}).$$

3. $$\int \frac{dx}{(2ax-x^2)^{\frac{3}{2}}} = \frac{x-a}{a^2\sqrt{2ax-x^2}}.$$

4. $$\int (x+\sqrt{1+x^2})^n dx = \frac{(x+\sqrt{1+x^2})^{n+1}}{2(n+1)} + \frac{(x+\sqrt{1+x^2})^{n-1}}{2(n-1)}.$$

5. $$\int \frac{(x+1)dx}{(3-x)\sqrt{3-2x-x^2}} = 2\tan^{-1}\sqrt{\frac{1-x}{3+x}} - \frac{4}{\sqrt{3}}\tan^{-1}\sqrt{\frac{3-3x}{3+x}}$$

$$= \cos^{-1}\frac{x+1}{2} - \frac{2}{\sqrt{3}}\cos^{-1}\frac{2x}{3-x}.$$

6. $$\int \frac{\sqrt{x^2+a^2}}{3x+4a}dx = \frac{z^2+a^2}{6z} - \frac{4a}{9}\log z + \frac{5a}{9}\log\frac{3z-a}{z+3a},$$

where $$z = x+\sqrt{x^2+a^2}.$$

208. Integrable Cases. — The preceding articles include those forms of irrational integrals that can be rationalized. In general, integrals containing fractional powers of polynomials above the first degree — except the square root of polynomials of the second degree — cannot be rationalized, and cannot be integrated in terms of the elementary functions, that is, cannot be expressed in terms of algebraic, exponential, logarithmic, trigonometric, or anti-trigonometric functions.

Every integral may be regarded as defining a certain function. It has been shown in Art. 192 that if $f(x)$ is any continuous function of x, $\int f(x)dx$ is a function of x, which may be geometrically represented by an area bounded by the curve $y=f(x)$; but this cannot always be expressed in terms of the elementary functions.

CHAPTER XXIV

TRIGONOMETRIC FORMS READILY INTEGRABLE

209. It is to be noticed that any power of a trigonometric function may be integrated by Formula I., when accompanied by its differential.

Thus,

$$\int \sin^n x \cos x\, dx = \frac{\sin^{n+1} x}{n+1}, \quad \int \cos^n x \sin x\, dx = -\frac{\cos^{n+1} x}{n+1},$$

$$\int \tan^n x \sec^2 x\, dx = \frac{\tan^{n+1} x}{n+1}, \quad \int \cot^n x \operatorname{cosec}^2 x\, dx = -\frac{\cot^{n+1} x}{n+1},$$

$$\int \sec^n x \sec x \tan x\, dx = \frac{\sec^{n+1} x}{n+1},$$

$$\int \operatorname{cosec}^n x \operatorname{cosec} x \cot x\, dx = -\frac{\operatorname{cosec}^{n+1} x}{n+1}.$$

Having in mind these integrals, the student should readily understand the transformations in the following articles.

210. **To find $\int \sin^n x\, dx$ or $\int \cos^n x\, dx$.** When n is an odd positive integer, we may integrate as in the following examples:

$$\int \sin^5 x\, dx = \int \sin^4 x \sin x\, dx = \int (1 - \cos^2 x)^2 \sin x\, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x) \sin x\, dx = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}.$$

Another example is

$$\int \cos^3 2x\,dx = \int \cos^2 2x \cos 2x\,dx = \frac{1}{2}\int (1-\sin^2 2x)\cos 2x\, 2\,dx$$

$$= \frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right).$$

211. To find $\int \sin^m x \cos^n x\,dx$. When either m or n is an odd positive integer, this form may be integrated in the same manner as in the preceding article. For example,

$$\int \sin^4 x \cos^5 x\,dx = \int \sin^4 x \cos^4 x \cos x\,dx = \int \sin^4 x (1-\sin^2 x)^2 \cos x\,dx$$

$$= \int (\sin^4 x - 2\sin^6 x + \sin^8 x)\cos x\,dx = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9}.$$

Another example is

$$\int \sin^3 x \cos^{\frac{3}{2}} x\,dx = \int \cos^{\frac{3}{2}} x \sin^2 x \sin x\,dx = \int \cos^{\frac{3}{2}} x (1-\cos^2 x)\sin x\,dx$$

$$= \int (\cos^{\frac{3}{2}} x - \cos^{\frac{7}{2}} x)\sin x\,dx = -\frac{2\cos^{\frac{5}{2}} x}{5} + \frac{2\cos^{\frac{9}{2}} x}{9}.$$

EXAMPLES

1. $\int \sin^7 x\,dx = -\cos x + \cos^3 x - \frac{3\cos^5 x}{5} + \frac{\cos^7 x}{7}.$

2. $\int \cos^9 x\,dx = \sin x - \frac{4\sin^3 x}{3} + \frac{6\sin^5 x}{5} - \frac{4\sin^7 x}{7} + \frac{\sin^9 x}{9}.$

3. $\int \sin^5 \frac{x}{2}\,dx = -2\cos\frac{x}{2} + \frac{4}{3}\cos^3\frac{x}{2} - \frac{2}{5}\cos^5\frac{x}{2}.$

4. $\int \sin^8\phi \cos^7\phi\,d\phi = \frac{\sin^9\phi}{9} - \frac{3\sin^{11}\phi}{11} + \frac{3\sin^{13}\phi}{13} - \frac{\sin^{15}\phi}{15}.$

5. $$\int \sin^5 2\,\theta \cos^3 2\,\theta\, d\theta = \frac{\sin^6 2\,\theta}{12} - \frac{\sin^8 2\,\theta}{16}.$$

6. $$\int (\sin^6 x + \cos^6 x) \sin^3 x \cos^2 x\, dx$$
$$= -\frac{\cos^3 x}{3} + \frac{4 \cos^5 x}{5} - \frac{6 \cos^7 x}{7} + \frac{\cos^9 x}{3}.$$

7. $$\int (\cos^3 \phi + \sin^3 \phi)(\cos^2 \phi - \sin^2 \phi)\, d\phi$$
$$= \sin^2 \phi \cos \phi + \cos^2 \phi \sin \phi + \frac{2}{5}(\sin^5 \phi + \cos^5 \phi).$$

8. $$\int \frac{\sin^7 y\, dy}{\cos y} = \frac{\cos^6 y}{6} - \frac{3 \cos^4 y}{4} + \frac{3 \cos^2 y}{2} + \log \sec y.$$

9. $$\int \frac{\cos^5 x\, dx}{\sin^4 x} = \sin x + \frac{2}{\sin x} - \frac{1}{3 \sin^3 x}.$$

10. $$\int \frac{\cos^3 x\, dx}{\sqrt{\sin^3 x}} = -\frac{2 \sin^2 x + 6}{3\sqrt{\sin x}}.$$

11. $$\int (\sin^m x \cos^3 x - \cos^m x \sin^3 x)\, dx$$
$$= \frac{\sin^{m+1} x + \cos^{m+1} x}{m+1} - \frac{\sin^{m+3} x + \cos^{m+3} x}{m+3}.$$

12. $$\int (\sin 2\,x + \cos 2\,x) \cos^3 x\, dx = \frac{2}{5}(\sin^5 x - \cos^5 x) + \sin x \cos^2 x.$$

13. $$\int \sin 4\,x \sin^3 x\, dx = \frac{4 \sin^5 x}{5} - \frac{8 \sin^7 x}{7}.$$

212. **To find $\int \tan^n x\,dx$, or $\int \cot^n x\,dx$.**

These forms can be readily integrated when n is any integer.

$$\int \tan^n x\,dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$$

Thus $\int \tan^n x\,dx$ is made to depend upon $\int \tan^{n-2} x\,dx$, and ultimately, by successive reductions, upon $\int \tan x\,dx$ or $\int dx$.

For example, $\int \tan^5 x\,dx = \int \tan^3 x\,(\sec^2 x - 1)\,dx$

$$= \frac{\tan^4 x}{4} - \int \tan^3 x\,dx.$$

$$\int \tan^3 x\,dx = \int \tan x\,(\sec^2 x - 1)\,dx$$

$$= \frac{\tan^2 x}{2} - \log \sec x.$$

Hence $$\int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x.$$

Another example is

$$\int \cot^6 x\,dx = \int \cot^4 x\,(\operatorname{cosec}^2 x - 1)\,dx = -\frac{\cot^5 x}{5} - \int \cot^4 x\,dx$$

$$= -\frac{\cot^5 x}{5} - \int \cot^2 x\,(\operatorname{cosec}^2 x - 1)\,dx = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} + \int \cot^2 x\,dx$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} + \int (\operatorname{cosec}^2 x - 1)\,dx = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x.$$

213. **To find $\int \sec^n x\,dx$ or $\int \operatorname{cosec}^n x\,dx$.** When n is an even positive integer, we may integrate as follows:

$$\int \sec^6 x\,dx = \int (\tan^2 x+1)^2 \sec^2 x\,dx = \int (\tan^4 x + 2\tan^2 x + 1)\sec^2 x\,dx$$

$$= \frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + \tan x.$$

Another example is

$$\int \operatorname{cosec}^4 x\,dx = \int (\cot^2 x+1)\operatorname{cosec}^2 x\,dx = -\frac{\cot^3 x}{3} - \cot x.$$

214. **To find $\int \tan^m x \sec^n x\,dx$ or $\int \cot^m x \operatorname{cosec}^n x\,dx$.** When n is an even positive integer, these forms may be integrated in the same manner as in the preceding article. For example,

$$\int \tan^6 x \sec^4 x\,dx = \int \tan^6 x\,(\tan^2 x+1)\sec^2 x\,dx$$

$$= \int (\tan^8 x + \tan^6 x)\sec^2 x\,dx = \frac{\tan^9 x}{9} + \frac{\tan^7 x}{7}.$$

When m is an odd positive integer, we may integrate as follows

$$\int \tan^5 x \sec^3 x\,dx = \int \tan^4 x \sec^2 x \sec x \tan x\,dx$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x\,dx$$

$$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x)\sec x \tan x\,dx$$

$$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3}.$$

Another example is

$$\int \cot^3 x \operatorname{cosec}^5 x\,dx = \int \cot^2 x \operatorname{cosec}^4 x \operatorname{cosec} x \cot x\,dx$$

$$= \int (\operatorname{cosec}^6 x - \operatorname{cosec}^4 x)\operatorname{cosec} x \cot x\,dx = -\frac{\operatorname{cosec}^7 x}{7} + \frac{\operatorname{cosec}^5 x}{5}.$$

EXAMPLES

1. $\int \tan^8 x\,dx = \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \tan x + x.$

2. $\int \cot^7 \frac{x}{2}\,dx = -\frac{1}{3}\cot^6 \frac{x}{2} + \frac{1}{2}\cot^4 \frac{x}{2} - \cot^2 \frac{x}{2} - \log \sin^2 \frac{x}{2}.$

3. $\int \sec^{10} y\,dy = \frac{\tan^9 y}{9} + \frac{4\tan^7 y}{7} + \frac{6\tan^5 y}{5} + \frac{4\tan^3 y}{3} + \tan y.$

4. $\int \operatorname{cosec}^8 3x\,dx = -\frac{1}{3}\left(\frac{\cot^7 3x}{7} + \frac{3\cot^5 3x}{5} + \cot^3 3x + \cot 3x\right).$

5. $\int (\sec x - \tan x)\sec^5 x \tan^4 x\,dx$

$$= \frac{1}{9}(\tan^9 x - \sec^9 x) + \frac{2}{7}(\tan^7 x + \sec^7 x) + \frac{1}{5}(\tan^5 x - \sec^5 x)$$

6. $\int (\sec^3 \phi + \tan^3 \phi)^2\,d\phi$

$$= \frac{2}{5}(\tan^5 \phi + \sec^5 \phi) + \frac{\tan^3 \phi}{3} - \frac{2\sec^3 \phi}{3} + 2\tan \phi - \phi$$

7. $\int \frac{\tan^7 x + 1}{\tan x + 1}\,dx = \frac{\tan^5 x}{5} - \frac{\tan^4 x}{4} + \tan x + \log \cos x.$

8. $\int \frac{\sec^5 x + \tan^5 x}{\sec x + \tan x}\,dx = \tan^3 x - \frac{2\sec^3 x}{3} + \sec x + x.$

9. $\int \frac{\sec^6 x + \sec^4 x}{\tan^3 x}\,dx = \frac{\tan^2 x}{2} - \cot^2 x + 3\log \tan x.$

10. $\int \frac{\operatorname{cosec}^6 \theta + \cot^6 \theta}{\operatorname{cosec}^2 \theta \cot^3 \theta}\,d\theta = \frac{1}{2}(\tan^2 \theta - \sin^2 \theta) + \log(\sin \theta \tan \theta).$

11. $\int\sqrt{\sec^3 x\tan x}\,(\sqrt{\sec^5 x}-\sqrt{\tan^5 x})\,dx$

$$=\frac{2}{7}\,(\tan^{\frac{7}{2}}x-\sec^{\frac{7}{2}}x)+\frac{2}{3}\,(\tan^{\frac{3}{2}}x+\sec^{\frac{3}{2}}x).$$

12. $\int(\sec^m x\tan^5 x-\tan^{m-1}x\,\sec^6 x)\,dx$

$$=\frac{\sec^{m+4}x-\tan^{m+4}x}{m+4}-\frac{2(\sec^{m+2}x+\tan^{m+2}x)}{m+2}+\frac{\sec^m x-\tan^m x}{m}.$$

A term $\int\sec^m x\operatorname{cosec}^n x\,dx$ may be integrated, when $m+n$ is even, by substituting $\operatorname{cosec}x=\frac{\sec x}{\tan x}$.

13. $\int\sec^5 x\operatorname{cosec}^3 x\,dx=\frac{\tan^4 x}{4}+\frac{3\tan^2 x}{2}-\frac{\cot^2 x}{2}+3\log\tan x.$

14. $\int(\sec^4 x-\operatorname{cosec}^2 x)^2\,dx$

$$=\frac{\tan^7 x}{7}+\frac{3\tan^5 x}{5}+\frac{\tan^3 x}{3}-3\tan x-\frac{\cot^3 x}{3}+\cot x.$$

215. To find $\int\sin^m x\cos^n x\,dx$ by Multiple Angles. The integration of this form, when either m or n is an odd positive integer, has been given in Art. 211. The following method is applicable when both m and n are even positive integers.

By trigonometric transformation $\sin^m x\cos^n x$, when m and n are positive integers, can be expressed in a series of terms of the first degree, involving sines and cosines of multiples of x.

If we use the method of Art. 211 for integrating terms with one odd exponent occurring during the process, the following formulæ for the double angle will be sufficient for the transformation of the terms with even exponents:

$$\sin A \cos A = \frac{1}{2}\sin 2A,$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A),$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A).$$

For example, required $\int \sin^4 x \cos^2 x \, dx.$

$$\sin^4 x \cos^2 x = (\sin x \cos x)^2 \sin^2 x = \frac{1}{8}\sin^2 2x(1 - \cos 2x)$$

$$= -\frac{1}{8}\sin^2 2x \cos 2x + \frac{1}{16}(1 - \cos 4x).$$

Hence $$\int \sin^4 x \cos^2 x \, dx = -\frac{\sin^3 2x}{48} + \frac{x}{16} - \frac{\sin 4x}{64}.$$

EXAMPLES

1. $\int \sin^4 x \, dx = \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right).$

2. $\int \cos^4 x \, dx = \frac{1}{4}\left(\frac{3x}{2} + \sin 2x + \frac{\sin 4x}{8}\right).$

3. $\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right).$

4. $\int \sin^6 x \, dx = \frac{1}{16}\left(5x - 4\sin 2x + \frac{\sin^3 2x}{3} + \frac{3}{4}\sin 4x\right).$

5. $\int \cos^6 x\,dx = \frac{1}{16}\left(5x + 4\sin 2x - \frac{\sin^3 2x}{3} + \frac{3}{4}\sin 4x\right).$

6. $\int \sin^4 x \cos^4 x\,dx = \frac{1}{128}\left(3x - \sin 4x + \frac{\sin 8x}{8}\right).$

7. $\int \cos^6 x \sin^2 x\,dx = \frac{1}{128}\left(5x + \frac{8}{3}\sin^3 2x - \sin 4x - \frac{\sin 8x}{8}\right).$

8. $$\int \sin^8 x\,dx = \frac{1}{16}\left(\frac{35x}{8} - 4\sin 2x + \frac{2}{3}\sin^3 2x + \frac{7}{8}\sin 4x \right.$$
$$\left. + \frac{\sin 8x}{64}\right).$$

CHAPTER XXV

INTEGRATION BY PARTS. REDUCTION FORMULÆ

216. Integration by Parts. From the differential of a product

$$d(uv) = u\,dv + v\,du,$$

we have
$$uv = \int u\,dv + \int v\,du.$$

Hence
$$\int u\,dv = uv - \int v\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This formula expresses a method of integration, which is called *integration by parts.*

For example, let us apply it to

$$\int x \log x\,dx.$$

Let $u = \log x$, then $dv = x\,dx$;

whence $du = \dfrac{dx}{x}$, and $v = \dfrac{x^2}{2}\cdot$

Substituting in (1), we have

$$\int \log x \cdot x\,dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4}\cdot$$

Integration by parts may be regarded as a process, which begins by integrating as if a certain factor were constant.

Thus in (2), if in $\int \log x \cdot x\,dx$ we treat $\log x$ as if it were a constant factor, we obtain $\log x \cdot \frac{x^2}{2}$. From this we must subtract a new integral formed as indicated by the following connecting lines.

$$\int \log x \cdot x\,dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\frac{dx}{x}.$$

This method of remembering the process may be found useful.

Another example is $\int x \cos x\,dx.$

Assuming $u = \cos x$, we have

$$\int x \cos x\,dx = \cos x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}(-\sin x\,dx).$$

As the new integral contains a higher power of x than the original integral, nothing is gained by this application of the process.

But if we take $u = x$, we find

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx$$

$$= x \sin x + \cos x.$$

EXAMPLES

1. $\int x^3 \log x\,dx = \frac{x^4}{4}\left(\log x - \frac{1}{4}\right).$

2. $\int x(e^{ax} + e^{-ax})\,dx = \frac{x}{a}(e^{ax} - e^{-ax}) - \frac{1}{a^2}(e^{ax} + e^{-ax}).$

3. $\int x(\sin 3x - \cos 3x)\,dx = \left(-\frac{x}{3} + \frac{1}{9}\right)\sin 3x - \left(\frac{x}{3} + \frac{1}{9}\right)\cos 3x.$

4. $\int x\left(\log\frac{x}{2}+\sin\frac{x}{2}\right)dx=\frac{x^2}{2}\log\frac{x}{2}-\frac{x^2}{4}-2\,x\cos\frac{x}{2}+4\sin\frac{x}{2}.$

5. $\int x(e^{2x}-1)^2dx=e^{4x}\left(\frac{x}{4}-\frac{1}{16}\right)-e^{2x}\left(x-\frac{1}{2}\right)+\frac{x^2}{2}.$

6. $\int\log(ax+b)\,dx=\left(x+\frac{b}{a}\right)\log(ax+b)-x.$

7. $\int(x+1)\log(x+3)\,dx=\frac{x^2+2\,x-3}{2}\log(x+3)-\frac{x^2}{4}+\frac{x}{2}.$

8. $\int\sec^4\phi\log\sin\phi\,d\phi=\left(\frac{\tan^3\phi}{3}+\tan\phi\right)\log\sin\phi-\frac{\tan\phi}{3}-\frac{2\,\phi}{3}.$

9. $\int\frac{\log x}{\sqrt{3\,x-2}}\,dx=\frac{2}{3}\sqrt{3x-2}\left(\log x-2\right)+\frac{4\sqrt{2}}{3}\tan^{-1}\sqrt{\frac{3x-2}{2}}.$

10. $\int\frac{\log(x+2)}{(x+1)^2}\,dx=-\frac{\log(x+2)}{x+1}+\log\frac{x+1}{x+2}.$

11. $\int\tan^{-1}\frac{x}{a}\,dx=x\tan^{-1}\frac{x}{a}-\frac{a}{2}\log(x^2+a^2).$

12. $\int x^2\tan^{-1}x\,dx=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\log(x^2+1).$

13. $\int\sin^{-1}\frac{x}{a}\,dx=x\sin^{-1}\frac{x}{a}+\sqrt{a^2-x^2}.$

14. $\int(3\,x^2-1)\sin^{-1}x\,dx=(x^3-x)\sin^{-1}x-\frac{(1-x^2)^{\frac{3}{2}}}{3}.$

15. $$\int x\sin^3 x\,dx = x\left(\frac{\cos^3 x}{3}-\cos x\right)+\frac{\sin^3 x}{9}+\frac{2\sin x}{3}.$$

16. $$\int x(\sec^6 x-\tan^6 x)\,dx = x\tan^3 x+\frac{x^2}{2}-\frac{\tan^2 x}{2}+\log\sec x.$$

17. $$\int\frac{\log(e^x+1)}{e^x}\,dx = x-\frac{e^x+1}{e^x}\log(e^x+1).$$

18. $$\int\log(a+\sqrt{x^2+a^2})\,dx = x\log(a+\sqrt{x^2+a^2}) + a\log(x+\sqrt{x^2+a^2})-x.$$

In each of the following examples integration by parts must be applied successively.

19. $$\int x^3e^{-2x}\,dx = -\frac{e^{-2x}}{8}(4x^3+6x^2+6x+3).$$

20. $$\int(e^{2x}-x)^3\,dx = \frac{e^{6x}}{6}-\frac{3e^{4x}}{16}(4x-1)+\frac{3e^{2x}}{4}(2x^2-2x+1)-\frac{x^4}{4}.$$

21. $$\int x^{n-1}(\log x)^2\,dx = \frac{x^n}{n}\left[(\log x)^2-\frac{2\log x}{n}+\frac{2}{n^2}\right].$$

22. $$\int x^3\sin 2x\,dx = \left(\frac{3x^2}{4}-\frac{3}{8}\right)\sin 2x-\left(\frac{x^3}{2}-\frac{3x}{4}\right)\cos 2x.$$

23. $$\int x(\tan^{-1}x)^2\,dx = \frac{x^2+1}{2}(\tan^{-1}x)^2-x\tan^{-1}x+\frac{1}{2}\log(x^2+1).$$

24. $$\int x\log(x+a)\log(x-a)\,dx = \frac{x^2-a^2}{2}\log(x+a)\log(x-a) - \frac{(x+a)^2}{4}\log(x+a)-\frac{(x-a)^2}{4}\log(x-a)+\frac{x^2}{4}.$$

217. To find $\int e^{ax}\sin nx\,dx$, **and** $\int e^{ax}\cos nx\,dx$.

Integrating by parts, with $u = e^{ax}$,

$$\int e^{ax}\sin nx\,dx = -\frac{e^{ax}\cos nx}{n} + \frac{a}{n}\int e^{ax}\cos nx\,dx. \quad . \quad . \quad (1)$$

Integrating the same, with $u = \sin nx$,

$$\int e^{ax}\sin nx\,dx = \frac{e^{ax}\sin nx}{a} - \frac{n}{a}\int e^{ax}\cos nx\,dx. \quad . \quad . \quad . \quad (2)$$

We see that (1) and (2) are two equations containing the two required integrals, $\int e^{ax}\sin nx\,dx$ and $\int e^{ax}\cos nx\,dx$. Eliminating the latter, by multiplying (1) by n^2, and (2) by a^2, and adding, gives

$$(a^2 + n^2)\int e^{ax}\sin nx\,dx = e^{ax}(a\sin nx - n\cos nx);$$

hence $$\int e^{ax}\sin nx\,dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2 + n^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting this in (1) and transposing, gives

$$\frac{a}{n}\int e^{ax}\cos nx\,dx = \frac{e^{ax}(an\sin nx + a^2\cos nx)}{(a^2 + n^2)n};$$

hence $$\int e^{ax}\cos nx\,dx = \frac{e^{ax}(n\sin nx + a\cos nx)}{a^2 + n^2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

EXAMPLES

The student is advised to apply the *process* of Art. 217 to Exs. 1–4. For the remaining examples he may substitute the values of a and n in (3) and (4).

1. $$\begin{cases} \int e^{3x}\sin 5x\,dx = \frac{e^{3x}}{34}(3\sin 5x - 5\cos 5x), \\ \int e^{3x}\cos 5x\,dx = \frac{e^{3x}}{34}(5\sin 5x + 3\cos 5x). \end{cases}$$

2. $$\begin{cases} \displaystyle\int e^{-2x}\sin x\,dx = -\frac{e^{-2x}}{5}(2\sin x + \cos x), \\ \displaystyle\int e^{-2x}\cos x\,dx = \frac{e^{-2x}}{5}(\sin x - 2\cos x). \end{cases}$$

3. $$\int e^{ax}\sin ax\,dx = \frac{e^{ax}}{2a}(\sin ax - \cos ax).$$

4. $$\int e^{-\frac{x}{2}}\cos\frac{x}{3}dx = \frac{6e^{-\frac{x}{2}}}{13}\left(2\sin\frac{x}{3} - 3\cos\frac{x}{3}\right).$$

5. $$\int \frac{\sin 2x + \cos 2x}{e^{3x}}dx = -\frac{\sin 2x + 5\cos 2x}{13e^{3x}}.$$

6. $$\int (e^{2x} + \sin 2x)(e^{x} + \cos x)\,dx = \frac{e^{3x}}{3} + \frac{e^{2x}}{5}(\sin x + 2\cos x) + \frac{e^{x}}{5}(\sin 2x - 2\cos 2x) - \frac{2\cos^3 x}{3}.$$

7. $$\int e^{2x}\cos^2 3x\,dx = \frac{e^{2x}}{4} + \frac{e^{2x}}{40}(3\sin 6x + \cos 6x).$$

8. $$\int e^{x}\sin 2x\sin 3x\,dx = \frac{e^{x}}{4}\left[\sin x + \cos x - \frac{5\sin 5x + \cos 5x}{13}\right].$$

9. $$\int xe^{2x}\cos x\,dx = \frac{e^{2x}}{25}[5x(\sin x + 2\cos x) - 4\sin x - 3\cos x].$$

218. Reduction Formulæ for Binomial Algebraic Integrals. These are formulæ by which the integral,

$$\int x^{m}(a + bx^{n})^{p}\,dx,$$

may be made to depend upon a similar integral, with either m or p numerically diminished. There are four such formulæ, as follows:

$$\int x^m (a+bx^n)^p\,dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{(np+m+1)b}-\frac{(m-n+1)a}{(np+m+1)b}\int x^{m-n}(a+bx^n)^p\,dx, \quad . \quad (A)$$

$$\int x^m (a+bx^n)^p\,dx$$

$$=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}+\frac{npa}{np+m+1}\int x^m(a+bx^n)^{p-1}\,dx, \quad . \quad . \quad . \quad . \quad (B)$$

$$\int x^m (a+bx^n)^p\,dx$$

$$=\frac{x^{m+1}(a+bx^n)^{p+1}}{(m+1)a}-\frac{(np+n+m+1)b}{(m+1)a}\int x^{m+n}(a+bx^n)^p\,dx, \quad . \quad (C)$$

$$\int x^m (a+bx^n)^p\,dx$$

$$=-\frac{x^{m+1}(a+bx^n)^{p+1}}{n(p+1)a}+\frac{np+n+m+1}{n(p+1)a}\int x^m(a+bx^n)^{p+1}\,dx. \quad . \quad (D)$$

Formulæ (*A*) and (*B*) are used when the exponent to be reduced, m or p, is positive, (*A*) changing m into $m-n$, and (*B*) changing p into $p-1$.

Formulæ (*C*) and (*D*) are used when the exponent to be reduced, m or p, is negative, (*C*) changing m into $m+n$, and (*D*) changing p into $p+1$.

If, in the application of one of these formulæ to a particular case, any denominator becomes zero, the formula is then inapplicable. For this reason,

Formulæ (*A*) and (*B*) fail, when $np+m+1=0$.
Formula (*C*) fails, when $m+1=0$.
Formula (*D*) fails, when $p+1=0$.

In these exceptional cases the required integral can be obtained without the use of reduction formulæ.

219. Derivation of Formula (*A*). Let us put for brevity

$$X = a + bx^n, \quad dX = nbx^{n-1}\,dx.$$

Then
$$\int x^m X^p\,dx = \int x^{m-n+1}\frac{X^p dX}{nb}.$$

Integrating by parts with $u = x^{m-n+1}$, we have

$$\int x^m X^p\,dx = \frac{x^{m-n+1}X^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)}\int x^{m-n}X^{p+1}\,dx. \quad . \quad . \quad (1)$$

Comparing the integrals in (1), we see that not only is m diminished by n, but p is increased by 1.

In order that p may remain unchanged, further transformation is necessary.

By substituting $X^{p+1} = (a + bx^n)X^p$,
the last integral may be separated into two.

$$\int x^{m-n}X^{p+1}\,dx = a\int x^{m-n}X^p\,dx + b\int x^m X^p\,dx.$$

Substituting this in (1) and freeing from fractions,

$$nb(p+1)\int x^m X^p\,dx = x^{m-n+1}X^{p+1}$$
$$-(m-n+1)\left(a\int x^{m-n}X^p\,dx + b\int x^m X^p\,dx\right).$$

Transposing the last integral to the first number,

$$(np + m + 1)\,b\int x^m X^p\,dx = x^{m-n+1}X^{p+1} - (m-n+1)\,a\int x^{m-n}X^p\,dx, \quad (2)$$

which immediately gives (*A*).

220. Derivation of Formula (*B*). Integrating by parts with $u = X^p$, we have

$$\int x^m X^p\,dx = X^p\frac{x^{m+1}}{m+1} - \int\frac{x^{m+1}}{m+1}pX^{p-1}bnx^{n-1}\,dx$$

$$= \frac{x^{m+1}X^p}{m+1} - \frac{npb}{m+1}\int x^{m+n}X^{p-1}\,dx. \quad . \quad . \quad . \quad . \quad (1)$$

Comparing the integrals, we see that not only is p decreased by 1, but that m is increased by n.

To avoid the change in m, substitute in the last integral of (1)

$$bx^n = X - a$$

Also freeing from fractions,

$$(m+1)\int x^m X^p\,dx = x^{m+1}X^p - np\left(\int x^m X^p\,dx - a\int x^m X^{p-1}\,dx\right).$$

Transposing to the first member the last integral but one,

$$(np+m+1)\int x^m X^p\,dx = x^{m+1}X^p + npa\int x^m X^{p-1}\,dx, \quad . \quad . \quad . \quad (2)$$

which immediately gives (*B*).

221. Derivation of Formula (*C*). This may be obtained from (2), Art. 219, by transposing the two integrals, and replacing throughout, $m-n$ by m. This gives

$$(m+1)\,a\int x^m X^p\,dx = x^{m+1}X^{p+1} - (np+m+n+1)\,b\int x^{m+n}X^p\,dx,$$

from which we obtain (*C*).

222. Derivation of Formula (*D*). This may be obtained from (2), Art. 220, by transposing the two integrals, and replacing $p-1$ by p. This gives

$$n(p+1)a\int x^m X^p\,dx = -x^{m+1}X^{p+1} + (np+n+m+1)\int x^m X^{p+1}dx,$$

from which we obtain (*D*).

EXAMPLES

1. $\displaystyle\int\frac{x^2dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

Here $\displaystyle\int\frac{x^2dx}{\sqrt{a^2-x^2}}=\int x^2(a^2-x^2)^{-\frac{1}{2}}dx.$

Apply (*A*), making

$$m=2,\quad n=2,\quad p=-\frac{1}{2},\quad a=a^2,\quad b=-1.$$

$$\int x^2(a^2-x^2)^{-\frac{1}{2}}\,dx=\frac{x(a^2-x^2)^{\frac{1}{2}}}{-2}-\frac{a^2}{-2}\int(a^2-x^2)^{-\frac{1}{2}}\,dx$$

$$=-\frac{x}{2}(a^2-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

2. $\displaystyle\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log(x+\sqrt{a^2+x^2}).$

Apply (*B*), making

$$m=0,\quad n=2,\quad p=\frac{1}{2},\quad a=a^2,\quad b=1.$$

$$\int(a^2+x^2)^{\frac{1}{2}}dx=\frac{x}{2}(a^2+x^2)^{\frac{1}{2}}+\frac{a^2}{2}\int\frac{dx}{(a^2+x^2)^{\frac{1}{2}}}$$

$$=\frac{x}{2}(a^2+x^2)^{\frac{1}{2}}+\frac{a^2}{2}\log(x+\sqrt{a^2+x^2}).$$

3. $\displaystyle\int\frac{dx}{x^3\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}}{2a^2x^2}+\frac{1}{2a^3}\sec^{-1}\frac{x}{a}.$

Apply (C), making

$$m=-3,\quad n=2,\quad p=-\frac{1}{2},\quad a=-a^2,\quad b=1.$$

$$\int x^{-3}(x^2-a^2)^{-\frac{1}{2}}dx=\frac{x^{-2}(x^2-a^2)^{\frac{1}{2}}}{2a^2}+\frac{1}{2a^2}\int x^{-1}(x^2-a^2)^{-\frac{1}{2}}dx$$

$$=\frac{(x^2-a^2)^{\frac{1}{2}}}{2a^2x^2}+\frac{1}{2a^3}\sec^{-1}\frac{x}{a}.$$

4. $\displaystyle\int\frac{dx}{x(x^2-a^2)^{\frac{3}{2}}}=-\frac{1}{a^2\sqrt{x^2-a^2}}-\frac{1}{a^3}\sec^{-1}\frac{x}{a}.$

Apply (D), making

$$m=-1,\quad n=2,\quad p=-\frac{3}{2},\quad a=-a^2,\quad b=1.$$

$$\int x^{-1}(x^2-a^2)^{-\frac{3}{2}}dx=-\frac{x^0(x^2-a^2)^{-\frac{1}{2}}}{a^2}-\frac{1}{a^2}\int x^{-1}(x^2-a^2)^{-\frac{1}{2}}dx$$

$$=-\frac{1}{a^2(x^2-a^2)^{\frac{1}{2}}}-\frac{1}{a^3}\sec^{-1}\frac{x}{a}.$$

5. $\displaystyle\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

6. $\displaystyle\int\frac{x^2\,dx}{\sqrt{x^2-a^2}}=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right).$

7. $\displaystyle\int(a^2-x^2)^{\frac{3}{2}}dx=\frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}.$

8. $\displaystyle\int(x^2-a^2)^{\frac{3}{2}}dx=\frac{x}{8}(2x^2-5a^2)\sqrt{x^2-a^2}+\frac{3a^4}{8}\log\left(x+\sqrt{x^2-a^2}\right).$

9. $\int x^2\sqrt{a^2-x^2}\,dx=\frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}.$

10. $\int x^2\sqrt{x^2+a^2}\,dx=\frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2}-\frac{a^4}{8}\log\left(x+\sqrt{x^2+a^2}\right)$

11. $\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{a^2\sqrt{a^2-x^2}}.$

12. Derive the formula of reduction used in Case IV of Rational Fractions.

$$\int\frac{dx}{(x^2+a^2)^n}=\frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}}+(2n-3)\int\frac{dx}{(x^2+a^2)^{n-1}}\right]$$

13. $\int\frac{dx}{x^3(x^2-a^2)^{\frac{3}{2}}}=-\frac{3x^2-a^2}{2a^4x^2\sqrt{x^2-a^2}}-\frac{3}{2a^5}\sec^{-1}\frac{x}{a}.$

14. $\int\frac{dx}{x^4\sqrt{x^2+1}}=\frac{2x^2-1}{3x^3}\sqrt{x^2+1}.$

15. $\int\frac{x^2\,dx}{\sqrt{2ax-x^2}}=-\frac{x+3a}{2}\sqrt{2ax-x^2}+\frac{3a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}.$

Write $\int\frac{x^2\,dx}{\sqrt{2ax-x^2}}=\int\frac{x^{\frac{3}{2}}\,dx}{\sqrt{2a-x}}$, and apply (*A*) twice.

16. $\int\frac{dx}{x\sqrt{2ax-x^2}}=-\frac{\sqrt{2ax-x^2}}{ax}.$

17. $\int\sqrt{2ax-x^2}\,dx=\frac{x-a}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x-a}{a}.$

$$\text{or }=\frac{x-a}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}.$$

Write $\int \sqrt{2\,ax - x^2}\,dx = \int \sqrt{a^2 - (x-a)^2}\,dx$, and substitute in Ex. 5.

18. $$\int x\sqrt{2\,ax - x^2}\,dx = -\frac{3\,a^2 + ax - 2\,x^2}{6}\sqrt{2\,ax - x^2} + \frac{a^3}{2}\operatorname{vers}^{-1}\frac{x}{a}.$$

19. $$\int \frac{\sqrt{2\,ax - x^2}\,dx}{x} = \sqrt{2\,ax - x^2} + a\operatorname{vers}^{-1}\frac{x}{a}.$$

20. $$\int \frac{x^m\,dx}{\sqrt{2\,ax - x^2}} = -\frac{x^{m-1}\sqrt{2\,ax - x^2}}{m} + \frac{(2\,m-1)a}{m}\int \frac{x^{m-1}\,dx}{\sqrt{2\,ax - x^2}}.$$

21. $$\int \frac{dx}{x^m\sqrt{2\,ax - x^2}}$$
$$= -\frac{\sqrt{2\,ax - x^2}}{(2\,m-1)ax^m} + \frac{m-1}{(2\,m-1)a}\int \frac{dx}{x^{m-1}\sqrt{2\,ax - x^2}}.$$

22. $$\int x^m\sqrt{2\,ax - x^2}\,dx$$
$$= -\frac{x^{m-1}(2\,ax - x^2)^{\frac{3}{2}}}{m+2} + \frac{(2\,m+1)a}{m+2}\int x^{m-1}\sqrt{2\,ax - x^2}\,dx.$$

23. $$\int \frac{\sqrt{2\,ax - x^2}\,dx}{x^m}$$
$$= -\frac{(2\,ax - x^2)^{\frac{3}{2}}}{(2\,m-3)ax^m} + \frac{m-3}{(2\,m-3)a}\int \frac{\sqrt{2\,ax - x^2}\,dx}{x^{m-1}}.$$

223. **Trigonometric Reduction Formulæ.**—The methods explained in Arts. 211, 214, are applicable only in certain cases.

By means of the following formulæ,

$$\int \sin^m x \cos^n x\,dx, \int \tan^m x \sec^n x\,dx, \text{ and} \int \cot^m x \operatorname{cosec}^n x\,dx,$$

may be obtained for all integral values of m and n, by successive reduction.

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x \, dx. \quad (1)$$

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x \, dx. \quad . \ (2)$$

$$\int \sin^m x \cos^n x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x \, dx. \quad . \ . \ . \ . \ (3)$$

$$\int \sin^m x \cos^n x \, dx$$

$$= -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x \, dx. \quad . \ . \ (4)$$

$$\int \sin^m x \, dx = -\frac{\sin^{m-1} x \cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x \, dx. \quad . \ . \ . \ . \ . \ . \ (5)$$

$$\int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x \, dx. \quad . \ . \ . \ . \ . \ . \ . \ (6)$$

$$\int \tan^m x \sec^n x \, dx = \frac{\tan^{m-1} x \sec^n x}{m+n-1} - \frac{m-1}{m+n-1}\int \tan^{m-2} x \sec^n x \, dx. \quad (7)$$

$$\int \cot^m x \operatorname{cosec}^n x \, dx$$

$$= -\frac{\cot^{m-1} x \operatorname{cosec}^n x}{m+n-1} - \frac{m-1}{m+n-1}\int \cot^{m-2} x \operatorname{cosec}^n x \, dx. \quad . \ (8)$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx \ . \ . \ . \ . \ . \ . \ . \ . \ (9)$$

$$\int \operatorname{cosec}^n x\,dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx. \quad . \; . \; . \quad (10)$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

$$\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

224. Derivation of the Preceding Formulæ. — To derive (1), we integrate by parts with $u = \sin^{m-1} x$.

$$\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x\,dx.$$

$$\int \sin^{m-2} x \cos^{n+2} x\,dx = \int \sin^{m-2} x \cos^n x\,dx - \int \sin^m x \cos^n x\,dx.$$

Substituting this in the preceding equation, and freeing from fractions, we have

$$(m+n)\int \sin^m x \cos^n x\,dx$$
$$= -\sin^{m-1} x \cos^{n+1} x + (m-1)\int \sin^{m-2} x \cos^n x\,dx,$$

which gives (1).

To derive (2), integrate by parts with $u = \cos^{n-1} x$, and proceed as in the derivation of (1).

Formula (3) may be derived from (1) by transposing the integrals, and replacing $m-2$ by m.

Formula (4) may be derived from (2) by transposing the integrals, and replacing $n-2$ by n.

To derive (5), make $n=0$ in (1); and to derive (6), make $m=0$ in (2).

The derivation of (7), (8), (9), and (10) is left to the student. We have already derived (11) and (12) in Art. 212.

EXAMPLES

1. $$\int \sin^6 x\,dx = -\frac{\cos x}{2}\left(\frac{\sin^5 x}{3} + \frac{5}{12}\sin^3 x + \frac{5}{8}\sin x\right) + \frac{5x}{16}.$$

2. $$\int \operatorname{cosec}^5 x\,dx = -\frac{\cos x}{4}\left(\frac{1}{\sin^4 x} + \frac{3}{2\sin^2 x}\right) + \frac{3}{8}\log\tan\frac{x}{2}.$$

3. $$\int \sec^7 x\,dx = \frac{\sin x}{2\cos^2 x}\left(\frac{1}{3\cos^4 x} + \frac{5}{12\cos^2 x} + \frac{5}{8}\right)$$
$$+\frac{5}{16}\log(\sec x + \tan x).$$

4. $$\int \cos^8 x\,dx = \frac{\sin x}{8}\left(\cos^7 x + \frac{7}{6}\cos^5 x + \frac{35}{24}\cos^3 x + \frac{35}{16}\cos x\right) + \frac{35x}{128}.$$

5. $$\int \sin^4 x\cos^2 x\,dx = \frac{\cos x}{2}\left(\frac{\sin^5 x}{3} - \frac{\sin^3 x}{12} - \frac{\sin x}{8}\right) + \frac{x}{16}.$$

6. $$\int \frac{\cos^4 x}{\sin^5 x}dx = \frac{3\cos x - 4\cos^3 x}{4\sin^4 x} - \frac{3\cos x}{8\sin^2 x} + \frac{3}{8}\log\tan\frac{x}{2}.$$

7. $$\int \frac{dx}{\sin^4 x\cos^3 x} = -\frac{1}{\cos^2 x}\left(\frac{1}{3\sin^3 x} + \frac{5}{3\sin x} - \frac{5}{2}\sin x\right)$$
$$+\frac{5}{2}\log(\sec x + \tan x).$$

8. $$\int \tan^4 x\sec^3 x\,dx = \left(\frac{\tan^3 x}{6} - \frac{\tan x}{8}\right)\sec^3 x + \frac{\sec x\tan x}{16}$$
$$+\frac{1}{16}\log(\sec x + \tan x).$$

9. $$\int \cot^2 x\operatorname{cosec}^5 x\,dx = \frac{\cot x\operatorname{cosec} x}{2}\left(-\frac{\operatorname{cosec}^4 x}{3} + \frac{\operatorname{cosec}^2 x}{12} + \frac{1}{8}\right)$$
$$-\frac{1}{16}\log\tan\frac{x}{2}.$$

CHAPTER XXVI

INTEGRATION BY SUBSTITUTION

225. The substitution of a new variable has been used in Chapter XXIII, for the rationalization of certain irrational integrals. We shall consider in this chapter some other cases where, by a change of variable, a given integral may be made to depend upon a new integral of simpler form.

We shall first consider some substitutions applicable to integrals of algebraic functions, and afterward those applicable to integrals of trigonometric functions.

226. Integrals of form $\int f(x^2)x\,dx$, containing $(a+bx^2)^{\frac{p}{q}}$. One of the most obvious substitutions, when applicable, is $x^2 = z$.

By this, any integral of the form $$\int f(x^2)x\,dx$$

is changed into $$\frac{1}{2}\int f(z)\,dz.$$

Integrals containing $(a+bx^2)^{\frac{p}{q}}$ are often of this form.

Take for example $$\int\frac{x^3\,dx}{\sqrt{1-x^2}}.$$

By the substitution $$x^2 = z,$$

$$\int\frac{x^3\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\int\frac{z\,dz}{\sqrt{1-z}}.$$

This is of the form of Art. 203, and is rationalized by putting $1-z=w^2$.

The two substitutions in succession are equivalent to the single substitution $1 - x^2 = w^2$.

Applying this to the given integral,

$$x^2 = 1 - w^2, \quad x\,dx = -w\,dw.$$

$$\int \frac{x^3\,dx}{\sqrt{1-x^2}} = -\int \frac{(1-w^2)w\,dw}{w} = -\int (1-w^2)\,dw$$

$$= -\left(w - \frac{w^3}{3}\right) = -\frac{w}{3}(3-w^2) = -\frac{\sqrt{1-x^2}}{3}(x^2+2).$$

EXAMPLES

1. $\displaystyle\int \frac{x^5\,dx}{\sqrt{2x^2+1}} = \frac{3x^4 - 2x^2 + 2}{30}\sqrt{2x^2+1}.$

2. $\displaystyle\int x^3(a^2-x^2)^{\frac{1}{5}}\,dx = \frac{5}{132}(6x^4 - a^2x^2 - 5a^4)(a^2-x^2)^{\frac{6}{5}}.$

3. $\displaystyle\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{2a}\log\frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+a^2}+a} = \frac{1}{2a}\log\frac{x^2}{(\sqrt{x^2+a^2}+a)^2}$

$$= \frac{1}{a}\log\frac{x}{\sqrt{x^2+a^2}+a}.$$

4. $\displaystyle\int \frac{x\,dx}{\sqrt[3]{x^2+1}-1} = \frac{3}{2}\left[\frac{(x^2+1)^{\frac{2}{3}}}{2} + (x^2+1)^{\frac{1}{3}} + \log(\sqrt[3]{x^2+1}-1)\right].$

5. $\displaystyle\int \frac{x\,dx}{x^2+2\sqrt{3-x^2}} = \frac{1}{4}\log(\sqrt{3-x^2}+1) + \frac{3}{4}\log(\sqrt{3-x^2}-3).$

227. Integration of Expressions containing $\sqrt{a^2-x^2}$ or $\sqrt{x^2 \pm a^2}$, by a Trigonometric Substitution. Frequently the shortest method of treating such integrals is to change the variable as follows:

For $\sqrt{a^2-x^2}$, let $x=a\sin\theta$ or $x=a\cos\theta$.

For $\sqrt{x^2+a^2}$, let $x=a\tan\theta$ or $x=a\cot\theta$.

For $\sqrt{x^2-a^2}$, let $x=a\sec\theta$ or $x=a\operatorname{cosec}\theta$.

For example, find $\displaystyle\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$.

Let $x=a\sin\theta$, $dx=a\cos\theta\,d\theta$,

$$a^2-x^2=a^2-a^2\sin^2\theta=a^2\cos^2\theta.$$

$$\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}=\int\frac{a\cos\theta\,d\theta}{a^3\cos^3\theta}=\frac{1}{a^2}\int\frac{d\theta}{\cos^2\theta}=\frac{\tan\theta}{a^2}=\frac{x}{a^2\sqrt{a^2-x^2}}.$$

Take for another example $\displaystyle\int\frac{dx}{x\sqrt{x^2+a^2}}$.

Let $x=a\tan\theta$.

$$\int\frac{dx}{x\sqrt{x^2+a^2}}=\int\frac{a\sec^2\theta\,d\theta}{a\tan\theta\cdot a\sec\theta}=\frac{1}{a}\int\frac{\sec\theta}{\tan\theta}d\theta=\frac{1}{a}\int\frac{d\theta}{\sin\theta}$$

$$=\frac{1}{a}\log(\operatorname{cosec}\theta-\cot\theta)=\frac{1}{a}\log\frac{\sqrt{x^2+a^2}-a}{x}.$$

Again, find $\displaystyle\int\frac{\sqrt{x^2-a^2}}{x}dx$.

Let $x=a\sec\theta$.

$$\int\frac{\sqrt{x^2-a^2}}{x}dx=\int\frac{a\tan\theta\cdot a\sec\theta\tan\theta\,d\theta}{a\sec\theta}=a\int\tan^2\theta\,d\theta$$

$$=a\int(\sec^2\theta-1)d\theta=a(\tan\theta-\theta)$$

$$=\sqrt{x^2-a^2}-a\sec^{-1}\frac{x}{a}.$$

EXAMPLES

1. $$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

2. $$\int\frac{dx}{x^2\sqrt{x^2+a^2}}=-\frac{\sqrt{x^2+a^2}}{a^2x}.$$

3. $$\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left(x+\sqrt{x^2-a^2}\right).$$

4. $$\int\frac{dx}{x^4\sqrt{a^2-x^2}}=-\frac{(2\,x^2+a^2)\sqrt{a^2-x^2}}{3\,a^4x^3}.$$

5. $$\int\frac{\sqrt{x^2+a^2}}{x^2}\,dx=-\frac{\sqrt{x^2+a^2}}{x}+\log\left(x+\sqrt{x^2+a^2}\right).$$

6. $$\int\frac{\sqrt{x^2-a^2}}{x^4}\,dx=\frac{(x^2-a^2)^{\frac{3}{2}}}{3a^2x^3}.$$

7. $$\int\frac{dx}{x^4\sqrt{x^2+1}}=\frac{(2\,x^2-1)\sqrt{x^2+1}}{3x^3}.$$

8. $$\int\sqrt{\frac{x+1}{x-1}}\,dx=\int\frac{x+1}{\sqrt{x^2-1}}\,dx=\sqrt{x^2-1}+\log\left(x+\sqrt{x^2-1}\right).$$

9. $$\int\frac{dx}{(x+1)\sqrt{x^2-1}}=\sqrt{\frac{x-1}{x+1}}.$$

10. $$\int\frac{dx}{(1-x)\sqrt{1-x^2}}=\sqrt{\frac{1+x}{1-x}}.$$

11. $$\int\frac{x\,dx}{\sqrt{3+2\,x-x^2}}=-\sqrt{3+2\,x-x^2}+\sin^{-1}\frac{x-1}{2}.$$

For $\sqrt{3+2x-x^2}=\sqrt{4-(x-1)^2}$, let $x-1=2\sin\theta$.

12. $$\int\frac{dx}{(x^2+2x+3)^{\frac{3}{2}}}=\frac{x+1}{2\sqrt{x^2+2x+3}}.$$

For $\sqrt{x^2+2x+3}=\sqrt{(x+1)^2+2}$, let $x+1=\sqrt{2}\tan\theta$.

13. $$\int\frac{dx}{(2ax-x^2)^{\frac{3}{2}}}=\frac{x-a}{a^2\sqrt{2ax-x^2}}.$$

14. $$\int\sqrt{\frac{x+5}{x+3}}\,dx=\sqrt{(x+5)(x+3)}+2\log\left(\sqrt{x+5}+\sqrt{x+3}\right).$$

15. $$\int\frac{x^2\,dx}{\sqrt{2ax-x^2}}=-\frac{(3a+x)\sqrt{2ax-x^2}}{2}+\frac{3a^2}{2}\sin^{-1}\frac{x-a}{a}.$$

228. Substitutions for the Integration of Trigonometric Functions. A trigonometric function can often be integrated by transforming it, by a change of variable, into an algebraic function. For this purpose two methods of substitution may be used, as shown in the two following articles.

229. Substitution, $\sin x=z$, $\cos x=z$, or $\tan x=z$.

Consider, for example, $\displaystyle\int\frac{\sin x\cos x\,dx}{1-\sin x+\cos^2 x}$.

Let $\sin x=z$, then $x=\sin^{-1}z$, $dx=\dfrac{dz}{\sqrt{1-z^2}}$.

$$\int\frac{\sin x\cos x\,dx}{1-\sin x+\cos^2 x}=\int\frac{z\sqrt{1-z^2}}{1-z+1-z^2}\,\frac{dz}{\sqrt{1-z^2}}=\int\frac{z\,dz}{2-z-z^2}$$

$$=\int\frac{z\,dz}{(2+z)(1-z)}=-\frac{2}{3}\int\frac{dz}{2+z}+\frac{1}{3}\int\frac{dz}{1-z}$$

$$=-\frac{2}{3}\log(2+z)-\frac{1}{3}\log(1-z)=-\frac{1}{3}\log\left[(2+\sin x)^2(1-\sin x)\right].$$

EXAMPLES

1. $$\int \frac{dx}{a^2+b^2\tan^2 x}=\frac{1}{a^2-b^2}\left[x-\frac{b}{a}\tan^{-1}\left(\frac{b}{a}\tan x\right)\right].$$

2. $$\int \frac{dx}{1+\tan x}=\frac{x}{2}+\frac{1}{2}\log(\sin x+\cos x).$$

3. $$\int \frac{\sin x}{\sin 4x}\,dx=\frac{1}{8}\log\frac{1-\sin x}{1+\sin x}+\frac{1}{4\sqrt{2}}\log\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}.$$

Let $\sin x=z$.

4. $$\int \frac{dx}{\sin x+\sin 2x}=\frac{1}{3}\log\frac{\sin x\,(1+\cos x)}{(1+2\cos x)^2}.$$ Let $\cos x=z$.

5. $$\int \frac{\sin x+\cos x}{\sin x+2\cos x}\,dx=\frac{3x}{5}-\frac{1}{5}\log(\sin x+2\cos x).$$

Let $\tan x=z$.

6. $$\int \frac{\tan x}{\tan 3x}\,dx=x+\frac{1}{\sqrt{3}}\log\frac{\tan x-\sqrt{3}}{\tan x+\sqrt{3}}.$$

7. Show, by transforming into algebraic functions, that only one of the following integrals can be expressed in terms of the elementary functions. (See Art. 208.)

$$\int \sqrt{\tan x}\,dx=\int\frac{\sqrt{z}\,dz}{1+z^2},$$ where $z=\tan x$.

$$\int \sqrt{\sin x}\,dx=\int\frac{\sqrt{z}\,dz}{\sqrt{1-z^2}}=\int\frac{z\,dz}{\sqrt{z-z^3}},$$ where $z=\sin x$.

230. The Rational Substitution, $\tan\frac{x}{2}=z$. By this substitution, $\sin x$, $\cos x$, $\tan x$, and dx are expressed rationally in terms of z. For

$$\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2z}{1+z^2}.$$

$$\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-z^2}{1+z^2}.$$

$$\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{2z}{1-z^2}.$$

From $$\frac{x}{2}=\tan^{-1}z,\quad dx=\frac{2\,dz}{1+z^2}.$$

It follows that the integral of any trigonometric function of x, not containing radicals, may be made to depend upon the integral of a rational function of z, and can therefore be expressed in terms of elementary functions of x.

231. To find $\int\frac{dx}{a+b\sin x}$. Applying the substitution of the preceding article, $\tan\frac{x}{2}=z$,

$$\int\frac{dx}{a+b\sin x}=\int\frac{\frac{2\,dz}{1+z^2}}{a+\frac{2\,bz}{1+z^2}}=\int\frac{2\,dz}{a(1+z^2)+2\,bz}$$

$$=\int\frac{2\,adz}{a^2z^2+2\,abz+a^2}=\int\frac{2\,adz}{(az+b)^2+a^2-b^2}.$$

If $a > b$, numerically,

$$\int\frac{dx}{a+b\sin x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{az+b}{\sqrt{a^2-b^2}}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}.$$

If $a < b$, numerically,

$$\int\frac{dx}{a+b\sin x}=\int\frac{2\,a\,dz}{(az+b)^2-(b^2-a^2)}=\frac{1}{\sqrt{b^2-a^2}}\log\frac{az+b-\sqrt{b^2-a^2}}{az+b+\sqrt{b^2-a^2}}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}.$$

232. To find $\int\frac{dx}{a+b\cos x}$.

$$\int\frac{dx}{a+b\cos x}=\int\frac{\frac{2\,dz}{1+z^2}}{a+\frac{b(1-z^2)}{1+z^2}}=\int\frac{2\,dz}{(a-b)z^2+a+b}$$

$$=\frac{2}{a-b}\int\frac{dz}{z^2+\frac{a+b}{a-b}}.$$

If $a > b$, numerically,

$$\int\frac{dx}{a+b\cos x}=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\frac{z\sqrt{a-b}}{\sqrt{a+b}}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right).$$

If $a < b$, numerically,

$$\int \frac{dx}{a+b\cos x} = -\frac{2}{b-a}\int \frac{dz}{z^2 - \dfrac{b+a}{b-a}}$$

$$= -\frac{1}{\sqrt{b^2-a^2}} \log \frac{z\sqrt{b-a} - \sqrt{b+a}}{z\sqrt{b-a} + \sqrt{b+a}}$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b-a}\tan\dfrac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\dfrac{x}{2} - \sqrt{b+a}}.$$

EXAMPLES

Integrate the following functions by means of the rational substitution.

1. $\displaystyle\int \frac{dx}{5-3\cos x} = \frac{1}{2}\tan^{-1}\left(2\tan\frac{x}{2}\right).$

2. $\displaystyle\int \frac{dx}{1+2\sin 2x} = \frac{1}{2\sqrt{3}}\log\frac{\tan x + 2 - \sqrt{3}}{\tan x + 2 + \sqrt{3}}.$

3. $\displaystyle\int \frac{dx}{5\sin x + 12\cos x} = \frac{1}{13}\log\frac{3\tan\dfrac{x}{2} + 2}{2\tan\dfrac{x}{2} - 3}.$

4. $\displaystyle\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}}\log\frac{b\tan\dfrac{x}{2} - a + \sqrt{a^2+b^2}}{b\tan\dfrac{x}{2} - a - \sqrt{a^2+b^2}}.$

5. $\displaystyle\int \frac{dx}{\sin x + \operatorname{vers} x} = \log\frac{\tan\dfrac{x}{2}}{1+\tan\dfrac{x}{2}}.$

6. $$\int\frac{dx}{3-\sin x+2\cos x}=\tan^{-1}\frac{\tan\frac{x}{2}-1}{2}.$$

7. $$\int\frac{dx}{5+7\sin x-\cos x}=\frac{1}{5}\log\frac{3\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}.$$

8. $$\int\frac{dx}{(1+\sin x+\cos x)^2}=\frac{1}{2}\tan\frac{x}{2}-\frac{1}{1+\tan\frac{x}{2}}-\log\left(1+\tan\frac{x}{2}\right).$$

233. Miscellaneous Substitutions. Various substitutions applicable to certain cases will be suggested by experience.

The *reciprocal substitution*, $x=\frac{1}{z}$, may be mentioned as simplifying many integrals.

EXAMPLES

Apply the reciprocal substitution $x=\frac{1}{z}$ to Exs. 1–6.

1. $$\int\frac{\sqrt{x^2-a^2}}{x^4}dx=\frac{(x^2-a^2)^{\frac{3}{2}}}{3a^2x^3}.$$

2. $$\int\frac{dx}{x^2\sqrt{x^2+a^2}}=-\frac{\sqrt{x^2+a^2}}{a^2x}.$$

3. $$\int\frac{\sqrt{2ax-x^2}}{x^3}dx=-\frac{(2ax-x^2)^{\frac{3}{2}}}{3ax^3}.$$

4. $$\int\frac{dx}{x\sqrt{a^2\pm x^2}}=\frac{1}{a}\log\frac{x}{a+\sqrt{a^2\pm x^2}}.$$

5. $$\int\frac{(x-x^3)^{\frac{1}{3}}}{x^4}dx=-\frac{3(x-x^3)^{\frac{4}{3}}}{8x^4}.$$

6. $$\int\frac{dx}{x\sqrt{8x^2+2x-1}}=\sin^{-1}\frac{x-1}{3x}.$$

7. $$\int\frac{x^3dx}{(x+2)^4}=\frac{8}{3(x+2)^3}-\frac{6}{(x+2)^2}+\frac{6}{x+2}+\log(x+2).$$

Let $x+2=z$.

8. $$\int(x+a)(x+b)^n dx=\frac{(x+b)^{n+2}}{n+2}+(a-b)\frac{(x+b)^{n+1}}{n+1}.$$

Let $x+b=z$.

9. $$\int\frac{dx}{x(a+bx)^3}=\frac{1}{a^3}\left[\frac{b^2x^2}{2(a+bx)^2}-\frac{2bx}{a+bx}-\log\frac{a+bx}{x}\right].$$

Let $a+bx=xz$.

10. $$\int\frac{\sin x}{\tan(x+a)}dx=\sin x-\sin a\log\tan\frac{x+a}{2}.$$ Let $x+a=z$.

11. $$\int\frac{dx}{e^{3x}+e^{2x}+e^x}=-x-e^{-x}+\log\sqrt{e^{2x}+e^x+1}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2e^x+1}{\sqrt{3}}.$$

Let $e^x=z$.

12. $$\int\sqrt{\frac{a-x}{b+x}}dx=\sqrt{(a-x)(b+x)}+(a+b)\sin^{-1}\sqrt{\frac{x+b}{a+b}}.$$

Substitute $b+x=z^2$, and the integral takes the form of **Ex. 5**, p. 289.

13. $$\int\sqrt{\frac{x+a}{x+b}}dx$$

$$=\sqrt{(x+a)(x+b)}+(a-b)\log(\sqrt{x+a}+\sqrt{x+b}).$$

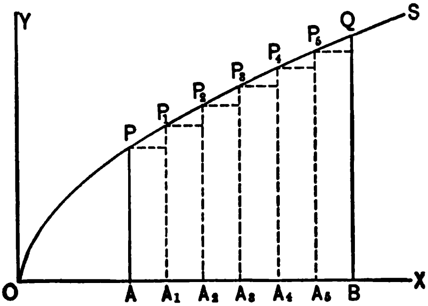
Substitute $x+b=z^2$, and the integral takes the form of **Ex. 2**, p. 288.

CHAPTER XXVII

INTEGRATION AS A SUMMATION. DEFINITE INTEGRAL

234. Integral the Limit of a Sum. An integral may be regarded and defined as the limit of a sum of a series of terms, and it is in this form that integration is most readily applied to practical problems.

235. Area of curve the limit of a sum of rectangles. Let it be required to find the area $PABQ$ included between the given curve OS, the axis of X, and the ordinates AP and BQ.



Let $y = x^{\frac{1}{2}}$ be the equation of the given curve.

Let $OA = a$, and $OB = b$.

Suppose AB divided into n equal parts (in the figure, $n = 6$), and let Δx denote one of the equal parts, AA_1, A_1A_2, $\cdots$

Then $AB = b - a = n\,\Delta x$.

At A_1, A_2, $\cdots$, draw the ordinates A_1P_1, A_2P_2, $\cdots$, and complete the rectangles PA_1, P_1A_2, $\cdots$

From the equation of the curve $y = x^{\frac{1}{2}}$,

$$PA = a^{\frac{1}{2}},\ P_1A_1 = (a + \Delta x)^{\frac{1}{2}},\ P_2A_2 = (a + 2\Delta x)^{\frac{1}{2}},\ \cdots,\ QB = b^{\frac{1}{2}}.$$

Area of rectangle $PA_1 = PA \times AA_1 = a^{\frac{1}{2}}\Delta x$.

Area of rectangle $P_1A_2 = P_1A_1 \times A_1A_2 = (a + \Delta x)^{\frac{1}{2}}\,\Delta x$.

Area of rectangle $P_2A_3 = P_2A_2 \times A_2A_3 = (a + 2\Delta x)^{\frac{1}{2}}\,\Delta x$.

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

Area of rectangle $P_5B = P_5A_5 \times A_5B = (b - \Delta x)^{\frac{1}{2}}\,\Delta x$.

The sum of all the n rectangles is

$$a^{\frac{1}{2}}\,\Delta x + (a+\Delta x)^{\frac{1}{2}}\,\Delta x + (a+2\,\Delta x)^{\frac{1}{2}}\,\Delta x + \cdots + (b-\Delta x)^{\frac{1}{2}}\,\Delta x,$$

which may be represented by $\sum_a^b x^{\frac{1}{2}}\,\Delta x$,

where $x^{\frac{1}{2}}\,\Delta x$ represents each term of the series, x taking in succession the values $a,\ a+\Delta x,\ a+2\,\Delta x,\ \cdots,\ b-\Delta x$.

It is evident that the area $PABQ$ is the limit of the sum of the rectangles, as n increases, and Δx decreases.

That is, $$\text{Area } PABQ = \text{Lim}_{\Delta x=0} \sum_a^b x^{\frac{1}{2}}\,\Delta x.$$

236. Definite Integral. From the preceding article

$$\sum_a^b x^{\frac{1}{2}}\,\Delta x = a^{\frac{1}{2}}\,\Delta x + (a+\Delta x)^{\frac{1}{2}}\,\Delta x + (a+2\,\Delta x)^{\frac{1}{2}}\,\Delta x + \cdots + (b-\Delta x)^{\frac{1}{2}}\,\Delta x.$$

The limit of this sum, as Δx approaches zero, is denoted by $\int_a^b x^{\frac{1}{2}}\,dx$. That is, by definition,

$$\int_a^b x^{\frac{1}{2}}\,dx = \text{Lim}_{\Delta x=0} \sum_a^b x^{\frac{1}{2}}\,\Delta x.$$

$\int_a^b x^{\frac{1}{2}}\,dx$ is called the *definite integral*, from a to b, of $x^{\frac{1}{2}}\,dx$.

It is to be noticed that a new definition is thus given to the symbol $\int$, which has been previously defined as an anti-differential. The relation between these two definitions will be shown in the following article.

237. Evaluation of the Definite Integral $\int_a^b x^{\frac{1}{2}}\,dx$. This is effected by finding a function whose derivative is $x^{\frac{1}{2}}$.

$$x^{\frac{1}{2}} = \frac{d}{dx}\left(\frac{2\,x^{\frac{3}{2}}}{3}\right).$$

By the definition of derivative, Art. 15,

$$\frac{d}{dx}\left(\frac{2\,x^{\frac{3}{2}}}{3}\right) = \text{Lim}_{\Delta x=0} \frac{\frac{2}{3}(x+\Delta x)^{\frac{3}{2}} - \frac{2\,x^{\frac{3}{2}}}{3}}{\Delta x} = x^{\frac{1}{2}}.$$

Hence
$$\frac{\frac{2}{3}(x+\Delta x)^{\frac{3}{2}}-\frac{2x^{\frac{3}{2}}}{3}}{\Delta x}=x^{\frac{1}{2}}+\epsilon,$$

where ϵ is a quantity that vanishes with Δx.

Hence
$$\frac{2}{3}(x+\Delta x)^{\frac{3}{2}}-\frac{2x^{\frac{3}{2}}}{3}=x^{\frac{1}{2}}\,\Delta x+\epsilon\,\Delta x.$$

Substituting in this equation successively for x,

$$a,\ a+\Delta x,\ a+2\,\Delta x,\ \cdots, b-\Delta x,$$

$$\frac{2}{3}(a+\Delta x)^{\frac{3}{2}}-\frac{2a^{\frac{3}{2}}}{3}=a^{\frac{1}{2}}\,\Delta x+\epsilon_1\,\Delta x,$$

$$\frac{2}{3}(a+2\,\Delta x)^{\frac{3}{2}}-\frac{2}{3}(a+\Delta x)^{\frac{3}{2}}=(a+\Delta x)^{\frac{1}{2}}\,\Delta x+\epsilon_2\Delta x,$$

$$\frac{2}{3}(a+3\,\Delta x)^{\frac{3}{2}}-\frac{2}{3}(a+2\,\Delta x)^{\frac{3}{2}}=(a+2\,\Delta x)^{\frac{1}{2}}\,\Delta x+\epsilon_3\,\Delta x,$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

$$\frac{2b^{\frac{3}{2}}}{3}-\frac{2}{3}(b-\Delta x)^{\frac{3}{2}}=(b-\Delta x)^{\frac{1}{2}}\Delta x+\epsilon_n\,\Delta x.$$

Adding and cancelling terms in the first members,

$$\frac{2b^{\frac{3}{2}}}{3}-\frac{2a^{\frac{3}{2}}}{3}=a^{\frac{1}{2}}\,\Delta x+(a+\Delta x)^{\frac{1}{2}}\Delta x+(a+2\,\Delta x)^{\frac{1}{2}}\Delta x+\cdots+(b-\Delta x)^{\frac{1}{2}}\Delta x$$
$$+\epsilon_1\,\Delta x+\epsilon_2\,\Delta x+\epsilon_3\,\Delta x+\cdots+\epsilon_n\,\Delta x$$
$$=\sum_a^b x^{\frac{1}{2}}\,\Delta x+\sum_a^b\epsilon\,\Delta x.\quad\ldots\ldots\ldots\ldots\quad(1)$$

Comparing with the figure, Art. 235, $\sum_a^b x^{\frac{1}{2}}\,\Delta x$, as we have before seen, represents the sum of the rectangles, and $\sum_a^b \epsilon\,\Delta x$ represents the sum of the triangular-shaped areas between these rectangles and the curve.

The latter sum approaches the limit zero, as Δx approaches zero.

For if ϵ_k is the greatest of the quantities $\epsilon_1, \epsilon_2, \cdots \epsilon_n$, it follows that

$$\sum_a^b \epsilon\,\Delta x < \epsilon_k \sum_a^b \Delta x,$$

that is,
$$\sum_a^b \epsilon\,\Delta x < \epsilon_k(b-a).$$

As ϵ_k vanishes with Δx,

$$\text{Lim}_{\Delta x=0} \sum_a^b \epsilon\,\Delta x = 0.$$

Taking the limit of (1),

$$\frac{2b^{\frac{3}{2}}}{3} - \frac{2a^{\frac{3}{2}}}{3} = \text{Lim}_{\Delta x=0} \sum_a^b x^{\frac{1}{2}}\,\Delta x,$$

that is,
$$\int_a^b x^{\frac{1}{2}}\,dx = \frac{2b^{\frac{3}{2}}}{3} - \frac{2a^{\frac{3}{2}}}{3} = \text{Area } PABQ.$$

Thus the value of $\int_a^b x^{\frac{1}{2}}\,dx$ is found from the integral

$$\int x^{\frac{1}{2}}dx = \frac{2x^{\frac{3}{2}}}{3},$$

by substituting for x, b and a in succession, thus giving

$$\frac{2b^{\frac{3}{2}}}{3} - \frac{2a^{\frac{3}{2}}}{3}.$$

The process may be expressed

$$\int_a^b x^{\frac{1}{2}}dx = \frac{2x^{\frac{3}{2}}}{3}\Bigg|_a^b = \frac{2b^{\frac{3}{2}}}{3} - \frac{2a^{\frac{3}{2}}}{3}.$$

This is called *integrating between limits*, the initial value a of the variable being the *lower limit*, and the final value b the *upper limit*.

In contradistinction

$$\int x^{\frac{1}{2}}\,dx = \frac{2x^{\frac{3}{2}}}{3} + C,$$

is called the *indefinite integral* of $x^{\frac{1}{2}}dx$.

238. General Definition of Definite Integral. In general if $f(x)$ is a given function of x which is continuous from a to b, inclusive, the definite integral

$$\int_a^b f(x)dx = \text{Lim}_{\Delta x=0}\Big[f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\,\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x\Big].$$

If $$\int f(x)\,dx = F(x), \quad \text{the indefinite integral,}$$

$$\int_a^b f(x)\,dx = F(b) - F(a). \quad \text{. . . . .} \quad (1)$$

This may be illustrated by the area bounded by a curve as in Art. 235, by supposing $y=f(x)$ to be the equation of the curve OS.

The proof of Art. 237 may be similarly generalized by substituting $f(x)$ for $x^{\frac{1}{2}}$, and $F(x)$ for $\frac{2\,x^{\frac{3}{2}}}{3}$.

Geometrically the definite integral $\int_a^b f(x)\,dx$ denotes the area swept over by the ordinate of a point of the curve $y=f(x)$, as x varies from a to b.

It is to be noticed that in Art. 192, by a somewhat different course of reasoning, we have arrived at the same result,

$$\text{Area } PABQ = F(b) - F(a).$$

239. Constant of Integration. It is to be noticed that the arbitrary constant C in the indefinite integral disappears in the definite integral.

Thus, if in evaluating $\int_a^b x^{\frac{1}{2}}\,dx$, we take for the indefinite integral

$$\int x^{\frac{1}{2}}\,dx = \frac{2\,x^{\frac{3}{2}}}{3} + C,$$

we find $$\int_a^b x^{\frac{1}{2}}\,dx = \frac{2\,b^{\frac{3}{2}}}{3} + C - \left(\frac{2\,a^{\frac{3}{2}}}{3} + C\right) = \frac{2\,b^{\frac{3}{2}}}{3} - \frac{2a^{\frac{3}{2}}}{3}.$$

Or if $$\int f(x)\,dx = F(x) + C,$$

$$\int_a^b f(x)\,dx = F(b) + C - [F(a) + C] = F(b) - F(a).$$

EXAMPLES

1. Compute $\sum_1^2 x^2\,\Delta x$ for different values of Δx.

When $\Delta x = .2$,

$$\sum_1^2 x^2\,\Delta x = (1^2 + \overline{1.2}^2 + \overline{1.4}^2 + \overline{1.6}^2 + \overline{1.8}^2)(.2) = 2.04.$$

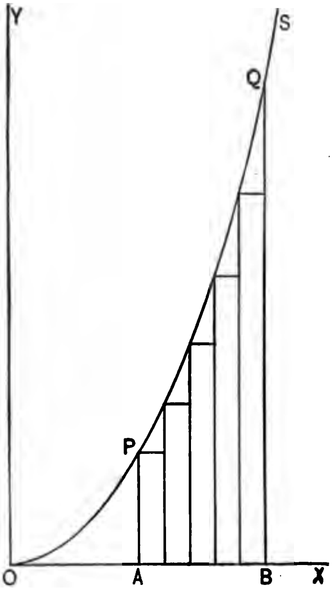
When $\Delta x = .1$,

$$\sum_1^2 x^2\Delta x = (1^2 + \overline{1.1}^2 + \overline{1.2}^2 + \cdots + \overline{1.9}^2)(.1) = 2.18.$$

When $\Delta x = .05$, $\sum_1^2 x^2\Delta x = 2.26$.

$$\text{Lim}_{\Delta x=0}\sum_1^2 x^2\,\Delta x = \int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{8-1}{3} = 2.33.$$

Curve OS, $y = x^2$. $OA = 1$, $OB = 2$. Area $PABQ = 2.33$ square units.



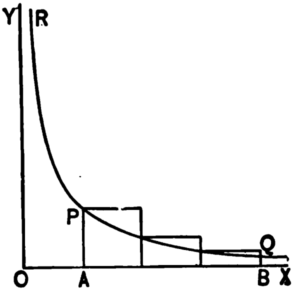
2. Compute $\sum_1^4 \frac{\Delta x}{x}$ for different values of Δx.

When $\Delta x = 1$,

$$\sum_1^4 \frac{\Delta x}{x} = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)(1) = 1.833.$$

When $\Delta x = .5$, $\sum_1^4 \frac{\Delta x}{x} = 1.593$.

When $\Delta x = .1$, $\sum_1^4 \frac{\Delta x}{x} = 1.426$.



$$\text{Lim}_{\Delta x=0}\sum_1^4\frac{\Delta x}{x}=\int_1^4\frac{dx}{x}=\log x\Big|_1^4=\log 4-\log 1=\log 4=1.386.$$

Curve RQ, $y=\frac{1}{x}\cdot$ $OA=1$, $OB=4$. Area $PABQ=1.386$ square units.

3. Compute $\sum_3^7 x\,\Delta x$, when $\Delta x=1$; when $\Delta x=.5$; when $\Delta x=.2$.
Ans. 18; 19; 19.6.

Find $\text{Lim}_{\Delta x=0}\sum_3^7 x\,\Delta x$. *Ans.* 20.

4. Compute $\sum_0^1\frac{\Delta x}{1+x^2}$, when $\Delta x=.2$; when $\Delta x=.1$; when $\Delta x=.05$. *Ans.* .833; .810; .798.

Find $\text{Lim}_{\Delta x=0}\sum_0^1\frac{\Delta x}{1+x^2}\cdot$ *Ans.* $\frac{\pi}{4}=.785$.

5. Compute $\sum_{10}^{13}\log_{10}x\,\Delta x$, when $\Delta x=1$; when $\Delta x=.5$; when $\Delta x=.3$. *Ans.* 3.121; 3.150; 3.161.

Find $\text{Lim}_{\Delta x=0}\sum_{10}^{13}\log_{10}x\,\Delta x$. *Ans.* $13\log_{10}13-3\log_{10}e-10=3.177$.

6. Compute $\sum_{\frac{\pi}{4}}^{\frac{\pi}{3}}\tan\phi\,\Delta\phi$, when $\Delta\phi=5°=\frac{\pi}{36}$; when $\Delta\phi=\frac{\pi}{60}$; when $\Delta\phi=\frac{\pi}{180}\cdot$ *Ans.* .316; .328; .340.

Find $\text{Lim}_{\Delta x=0}\sum_{\frac{\pi}{4}}^{\frac{\pi}{3}}\tan\phi\,\Delta\phi$. *Ans.* $\log_e\sqrt{2}=.346$.

7. $\int_2^3(x^2-4)^2x\,dx=\frac{125}{6}\cdot$

8. $\int_5^9\frac{x\,dx}{\sqrt{x^2+144}}=2.$

9. $\int_{\frac{a}{2}}^{a}\frac{dy}{\sqrt{a^2-y^2}}=\frac{\pi}{3}\cdot$

10. $\int_1^2\frac{dx}{x^2-x+1}=\frac{\pi}{3\sqrt{3}}\cdot$

11. $\int_0^2\frac{e^x\,dx}{e^{2x}+1}=\tan^{-1}e^2-\frac{\pi}{4}\cdot$

12. $\int_0^{\infty}\frac{8\,a^3\,dx}{x^2+4\,a^2}=2\,\pi a^2.$

13. $\displaystyle\int_2^3 \frac{dx}{\sqrt{(x-2)(3-x)}} = \pi.$ 14. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5\phi\, d\phi = \frac{8}{15}.$

15. $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2\theta\, d\theta = \frac{\pi}{12} - \frac{1}{8}.$ 16. $\displaystyle\int_0^{\frac{\pi}{6}} \text{vers}^2\, 2\,\theta\, d\theta = \frac{\pi}{4} - \frac{7\sqrt{3}}{16}.$

17. $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos 2\theta - \cos 2a}{\cos\theta - \cos a}\, d\theta = 1 + \frac{\pi}{3}\cos a.$

18. $\displaystyle\int_0^2 \frac{x^3\, dx}{(x+2)^3} = \frac{17}{4} - 6\log 2.$

19. $\displaystyle\int_0^1 \frac{dx}{1 + 2x + 2x^2 + 2x^3 + x^4} = \frac{1}{4}\log(2e).$

20. $\displaystyle\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx = \pi - 2.$

21. $\displaystyle\int_a^{2a} x\log(x+a)\, dx = \frac{3a^2}{2}\log(3a) - \frac{a^2}{4}.$

22. $\displaystyle\int_0^4 \tan^{-1}\frac{x}{4}\, dx = \pi - \log 4.$

23. $\displaystyle\int_a^b \frac{x^2\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a+b}\left(\frac{\pi}{4} - \tan^{-1}\frac{a}{b}\right).$

By (5) and (6), Art. 223, we find

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx,$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx;$$

from which derive the following results:

24. If n is even,

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \frac{\pi}{2}.$$

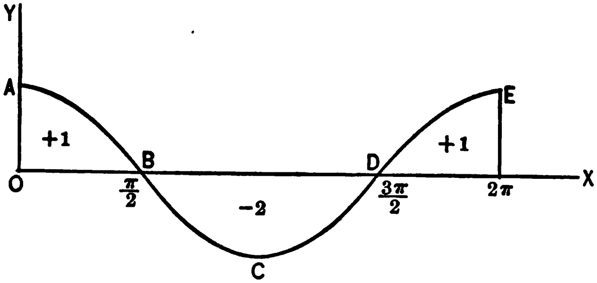
25. If n is odd,

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n}.$$

240. Sign of Definite Integral. In considering the definite integral $\int_a^b f(x)\, dx$, we have supposed $a < b$, and $f(x)$ to be positive between the limits a and b.

If $f(x)$ is negative from $x = a$ to $x = b$, $\sum_a^b f(x)\, \Delta x$, being the sum of a series of negative terms, is negative, and consequently $\int_a^b f(x)\, dx$ is negative.

If $f(x)$ changes sign between $x = a$ and $x = b$, $\int_a^b f(x)\, dx$ is the algebraic sum of a positive and a negative quantity.



For example, $\int_0^{\frac{\pi}{2}} \cos x\, dx = 1 = \text{area } AOB.$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx = -2 = \text{area } BCD.$$

$$\int_0^{\frac{3\pi}{2}} \cos x\,dx = -1 = 1 - 2.$$

$$\int_0^{2\pi} \cos x\,dx = 0 = 1 - 2 + 1.$$

The change of sign resulting from $a < b$ is considered in Art. 243.

241. Infinite Limits. In the definition of a definite integral the limits have been assumed to be finite. When one of the limits is infinite, the expression may be thus defined:

$$\int_a^\infty f(x)\,dx = \text{Lim}_{b=\infty} \int_a^b f(x)\,dx.$$

For example, consider Ex. 12, p. 312,

$$\int_0^\infty \frac{8\,a^3\,dx}{x^2 + 4\,a^2} = \text{Lim}_{b=\infty} \int_0^b \frac{8\,a^3\,dx}{x^2 + 4\,a^2} = \text{Lim}_{b=\infty} \left(4\,a^2 \tan^{-1} \frac{b}{2a}\right) = 2\pi\,a^2.$$

Referring to Art. 126, we find the geometrical interpretation of this result.

The area included between the curve, the axes of X and Y, and a variable ordinate, approaches the limit $2\pi a^2$, as the distance of the ordinate is indefinitely increased.

Applying the same process to $\int_1^\infty \frac{dx}{x}$, we find

$$\text{Lim}_{b=\infty} \int_1^b \frac{dx}{x} = \text{Lim}_{b=\infty} \log b = \infty.$$

In this case there is no limit, and the expression $\int_1^\infty \frac{dx}{x}$ has no meaning.

242. Infinite Values of $f(x)$. In the definition of $\int_a^b f(x)\,dx$, $f(x)$ is assumed to be a continuous function from $x = a$ to $x = b$. If $f(x)$ is continuous for all values from a to b except $x = a$, where it is infinite, the definite integral may be defined thus:

$$\int_a^b f(x)\,dx = \text{Lim}_{h=0} \int_{a+h}^b f(x)\,dx.$$

If $f(b)=\infty$, $f(x)$ being continuous for other values of x,

$$\int_a^b f(x)\,dx=\text{Lim}_{h=0}\int_a^{b-h} f(x)\,dx.$$

For example, consider Ex. 9, p. 312,

$$\int_{\frac{a}{2}}^{a}\frac{dy}{\sqrt{a^2-y^2}}.$$

Here $\dfrac{1}{\sqrt{a^2-y^2}}=\infty$, when $y=a$.

Hence

$$\int_{\frac{a}{2}}^{a}\frac{dy}{\sqrt{a^2-y^2}}=\text{Lim}_{h=0}\int_{\frac{a}{2}}^{a-h}\frac{dy}{\sqrt{a^2-y^2}}=\text{Lim}_{h=0}\left(\sin^{-1}\frac{a-h}{a}-\frac{\pi}{6}\right)$$

$$=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}.$$

Another example is Ex. 13, p. 313,

$$\int_2^3\frac{dx}{\sqrt{(x-2)(3-x)}}.$$

Here $\dfrac{1}{\sqrt{(x-2)(3-x)}}=\infty$, when $x=2$, and also when $x=3$.

Hence $\displaystyle\int_2^3\frac{dx}{\sqrt{(x-2)(3-x)}}=\text{Lim}_{h=0}\int_{2+h}^{3-h}\frac{dx}{\sqrt{(x-2)(3-x)}}$

$$=\text{Lim}_{h=0}\sin^{-1}(2x-5)\Big|_{2+h}^{3-h}=\text{Lim}_{h=0}[\sin^{-1}(1-2h)-\sin^{-1}(-1+2h)]$$

$$=\sin^{-1}1-\sin^{-1}(-1)=\pi.$$

If $f(x)$ is infinite for some value c between a and b, and is continuous for other values, the definite integral should be separated into two.

$$\int_a^b f(x)\,dx=\int_c^b f(x)\,dx+\int_a^c f(x)\,dx.\quad\text{See Art. 243.}$$

These new definite integrals may be treated as already explained.

EXAMPLES

1. $\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$.
2. $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$.
3. $\int_1^2 \frac{dx}{\sqrt{x^2-1}} = \log(2+\sqrt{3})$.
4. $\int_1^\infty \frac{dx}{x(1+x^2)} = \frac{1}{2}\log 2$.

243. Change of Limits. The sign of a definite integral is changed by the transposition of the limits,

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

This is evident from (1), Art. 238, and also from the definition. For if x varies from b to a, the sign of Δx is opposite to that where x varies from a to b. Hence the signs of all the terms of $\sum_a^b f(x)\,\Delta x$ will be changed, if the limits a and b are transposed.

$$\sum_b^a f(x)\,\Delta x = -\sum_a^b f(x)\,\Delta x;$$

Hence $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

A definite integral may be separated into two or more definite integrals by the relation,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

This follows directly from the definition.

244. Change of Limits for a Change of Variable. When a new variable is used in obtaining the indefinite integral, we may avoid returning to the original variable, by changing the limits to correspond with the new variable.

For example, to evaluate

$$\int_0^4 \frac{dx}{1+\sqrt{x}}, \quad \text{assume} \quad \sqrt{x} = z.$$

Then we have $$\frac{dx}{1+\sqrt{x}}=\frac{2\,z\,dz}{1+z}.$$

Now when $x=4$, $z=2$; and when $x=0$, $z=0$.

Hence $$\int_0^4\frac{dx}{1+\sqrt{x}}=\int_0^2\frac{2\,z\,dz}{1+z}=2\left[z-\log(1+z)\right]_0^2$$
$$=4-2\log 3.$$

EXAMPLES

1. $\displaystyle\int_2^7 x\sqrt{x+2}\,dx=\frac{886}{15}.$ Let $x+2=z^2$.

2. $\displaystyle\int_2^3 (x-2)^n\,x\,dx=\frac{3\,n+5}{n^2+3\,n+2}.$ Let $x-2=z$.

3. $\displaystyle\int_0^2\frac{x^3\,dx}{(x^2+1)^{\frac{2}{3}}}=\frac{3}{8}\,(3+\sqrt[3]{5}).$ Let $x^2+1=z^3$.

4. $\displaystyle\int_a^{2a}\sqrt{2\,ax-x^2}\,dx=\frac{\pi a^2}{4}.$ Let $x-a=a\sin\theta$.

5. $\displaystyle\int_2^3\frac{\sqrt{3+2\,x-x^2}}{(x-1)^2}\,dx=\sqrt{3}-\frac{\pi}{3}.$ Let $x=1+2\sin\theta$.

6. $\displaystyle\int_a^b\sqrt{(x-a)(b-x)}\,dx=\frac{\pi}{8}\,(b-a)^2.$ Let $x=a\cos^2\phi+b\sin^2\phi$.

7. $\displaystyle\int_0^a(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}\,dx=\frac{3\,\pi a^2}{32}.$ Let $x=a\sin^3\theta$.

8. $\displaystyle\int_{\frac{a}{2}}^a x^4\sqrt{a^2-x^2}\,dx=\left(\frac{\sqrt{3}}{64}+\frac{\pi}{48}\right)a^6.$ Let $x=a\sin\theta$.

245. Definite Integral as a Sum. In the application of integration it is often convenient, in forming the definite integral from the data of the problem, to regard $\int_a^b f(x)\,dx$ as the sum of an infinite number of infinitely small terms, $f(x)\,dx$ being called an *element* of the required definite integral.

From this point of view,

$$\int_a^b f(x)\,dx = f(a)\,dx + f(a+dx)\,dx + f(a+2\,dx)\,dx + \cdots + f(b)\,dx.$$

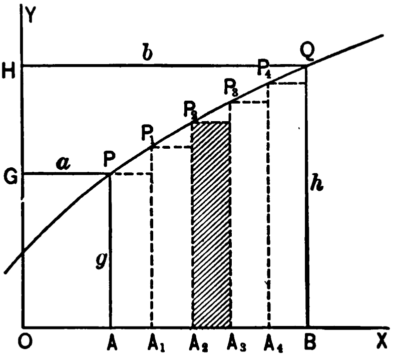
This may be regarded as an abbreviation of the definition of a definite integral given in Art. 238.

CHAPTER XXVIII

APPLICATION OF INTEGRATION TO PLANE CURVES. APPLICATION TO CERTAIN VOLUMES

246. Areas of Curves. Rectangular Coördinates. We have already used this problem as an illustration of a definite integral. We will now consider it more generally, and derive the formula for the area in rectangular coördinates.

247. To find the area between a given curve, the axis of X, and two given ordinates AP and BQ; that is, to find the area generated by the ordinate moving from AP to BQ.



Let $OA = a$, $OB = b$.

Let x and y be the coördinates of any point P_2 of the curve; then

$$x + \Delta x,\quad y + \Delta y,$$

will be the coördinates of P_3.

The area of the rectangle $P_2A_2A_3$ is

$$P_2A_2 \times A_2A_3 = y\ \Delta x.^*$$

The sum of all the rectangles PAA_1, $P_1A_1A_2$, $P_2A_2A_3$, $\cdots$, may be represented by $\sum_a^b y\ \Delta x$.

The required area $PQBA$ is the limit of the sum of the rectangles, as Δx is indefinitely diminished. That is

$$A = \int_a^b y\,dx,$$

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* By Art. 245, one readily sees that this rectangle represents an *element of area.*

the lower limit $a = OA$, being the initial value of x, and the upper limit $b = OB$, the final value of x.

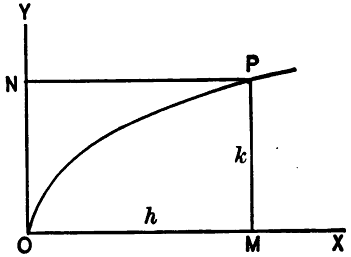
Similarly the area between the curve, the axis of Y, and two given abscissas, GP and HQ, is

$$A = \int_g^h x\,dy,$$

the limits of integration being the initial and final values of y, $g = OG$, and $h = OH$.

EXAMPLES

1. Find the area between the parabola $y^2 = 4\,ax$ and the axis of X, from the origin to the ordinate at the point (h, k).



Here $A = \int_0 y\,dx = \int_0^h 2\,a^{\frac{1}{2}}x^{\frac{1}{2}}\,dx$

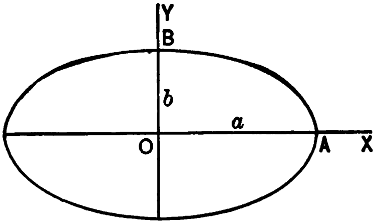
$$= \frac{4\,a^{\frac{1}{2}}x^{\frac{3}{2}}}{3}\Bigg|_0^h = \frac{4\,a^{\frac{1}{2}}h^{\frac{3}{2}}}{3}.$$ [*]

Since $k^2 = 4\,ah$, $k = 2\,a^{\frac{1}{2}}h^{\frac{1}{2}}$, which gives

$$A = \frac{2}{3}h\,2\,a^{\frac{1}{2}}h^{\frac{1}{2}} = \frac{2}{3}hk = \frac{2}{3}\,OMPN.$$

2. Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Area BOA

$$= \int_0^a y\,dx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\pi ab}{4}.$$

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[*] In finding areas, after the *element of area* and the *limits of integration* are chosen, the problem reduces to the evaluation of a definite integral.

The entire area $= \pi ab$.

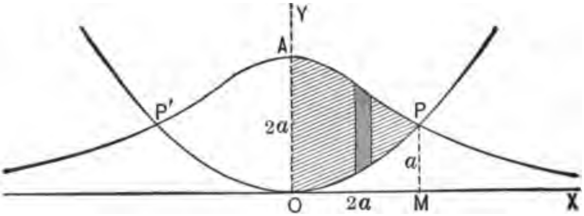
Or we may integrate by letting $x = a \sin \phi$.

$$\text{Then } \int_0^a \sqrt{a^2 - x^2}\, dx = a^2 \int_0^{\frac{\pi}{2}} \cos^2 \phi\, d\phi = \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\phi)\, d\phi = \frac{\pi a^2}{4}.$$

$$\text{Area } BOA = \frac{b}{a} \cdot \frac{\pi a^2}{4} = \frac{\pi ab}{4}.$$

3. Find the area included between the parabola $x^2 = 4ay$, and the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$. *Ans.* $\left(2\pi - \dfrac{4}{3}\right)a^2$.

Having found the point of intersection P, $(2a, a)$, we proceed as follows:



Area $AOP = AOMP - OMP$

$$= \int_0^{2a} \frac{8a^3 dx}{x^2 + 4a^2} - \int_0^{2a} \frac{x^2 dx}{4a} = \pi a^2 - \frac{2a^2}{3}. \quad *$$

Area between two curves $= \left(2\pi - \dfrac{4}{3}\right)a^2$.

4. Find the area of the parabola

$$(y - 5)^2 = 8(2 - x),$$

on the right of the axis of Y. *Ans.* $10\frac{2}{3}$.

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* Length of element of area is the y of the witch minus the y of the parabola.

5. Show that the area of a sector of the equilateral hyperbola $x^2 - y^2 = a^2$, included between the axis of X and a diameter through the point (x, y) of the curve, is $\frac{a^2}{2}\log\frac{x+y}{a}$.

6. Find the entire area within the curve (Art. 133) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

Ans. $\frac{3}{4}\pi ab$.

7. Find the entire area within the hypocycloid (Art. 132) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. Let $x = a\sin^3\phi$.

Ans. $\frac{3\pi a^2}{8}$.

8. Find the entire area between the cissoid (Art. 125) $y^2 = \frac{x^3}{2a - x}$ and the line $x = 2a$, its asymptote.

Ans. $3\pi a^2$.

9. Find the area of one loop of the curve (Art. 134) $a^4y^2 = a^2x^4 - x^6$.

Ans. $\frac{\pi a^2}{8}$.

Also from $x = \frac{a}{2}$ to $x = a$.

Ans. $\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right)\frac{a^2}{4}$.

10. Find the area of the evolute of the ellipse (Art. 167) $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

Ans. $\frac{3\pi}{8}\frac{(a^2-b^2)^2}{ab}$

11. What is the ratio between a and b, when the areas of the ellipse and its evolute are equal?

Ans. $\frac{a}{b} = \frac{\sqrt{5} + \sqrt{2}}{\sqrt{3}} \simeq 2.11$.

12. Find the area included between the parabolas

$y^2 = ax$ and $x^2 = by$.

Ans. $\frac{ab}{3}$

13. Find the area included between the parabola

$y^2 = 2x$ and the circle $y^2 = 4x - x^2$. *Ans.* $2\left(\pi - \frac{8}{3}\right)$

14. Find the area included between the parabola

$y^2 = 4ax$, and its evolute (Art. 167) $27ay^2 = 4(x - 2a)^3$.

Ans. $\frac{352\sqrt{2}}{15}a^2$.

Parametric Equations. Instead of a single equation between x and y for the equation of a curve, the relation between x and y may be expressed by means of a third variable. Thus the equations

$$x = a\sin\phi, \quad y = a\cos\phi, \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

represent a circle; for if we eliminate ϕ from (1) we have

$$x^2 + y^2 = a^2(\sin^2\phi + \cos^2\phi) = a^2.$$

Equations (1) are called the *parametric equations* of the circle, and the third variable ϕ is called the parameter.

The formula $A = \int y\,dx$ is applied to (1) by substituting

$$y = a\cos\phi, \quad dx = a\cos\phi\,d\phi.$$

For a quadrant of the circle

$$A = \int_0^a y\,dx = \int_0^{\frac{\pi}{2}} a^2\cos^2\phi\,d\phi = \frac{\pi a^2}{4}.$$

15. Find the area of one arch of the cycloid (Art. 284)

$x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$. *Ans.* $3\pi a^2$.

16. The parametric equations of the trochoid, described by a point at distance b from the centre of a circle, radius a, which rolls upon a straight line, are

$$x = a\phi - b\sin\phi, \quad y = a - b\cos\phi. \qquad \text{(Art. 292)}$$

Find the area of one arch of the trochoid above the tangent at the lowest points of the curve.

Ans. $\pi(2a + b)b$, when $b < a$ or $b > a$

248. Areas of Curves. Polar Coördinates. To find the Area POQ included between a Given Curve PQ and Two Given Radii Vectores; that is, to find the area generated by the radius vector turning from OP to OQ.

Let $POX = \alpha$, $QOX = \beta$.

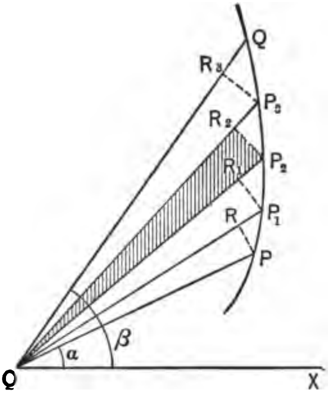
Let r and θ be the coördinates of any point P_2 of the curve, then

$$r + \Delta r, \quad \theta + \Delta\theta,$$

will be the coördinates of P_3.

The area of the circular sector P_2OR_2 is

$$\frac{1}{2} OP_2 \times P_2R_2 = \frac{1}{2} r \cdot r\,\Delta\theta = \frac{1}{2} r^2\,\Delta\theta.$$



The sum of the sectors POR, P_1OR_1, P_2OR_2, $\cdots$, may be represented by

$$\sum\nolimits_{\alpha}^{\beta} \frac{1}{2} r^2\,\Delta\theta.$$

The required area POQ is the limit of the sum of the sectors, as $\Delta\theta$ approaches zero. That is,

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta,$$

the initial value of θ, $\alpha = POX$, being the lower limit, and the final value of θ, $\beta = QOX$, the upper limit.

EXAMPLES

1. Find the area of one loop of the curve (Art. 144) $r = a \sin 2\,\theta$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2 \sin^2 2\,\theta\,d\theta = \frac{a^2}{4}\int_0^{\frac{\pi}{2}} (1 - \cos 4\,\theta)\,d\theta$$

$$= \frac{a^2}{4}\left(\theta - \frac{\sin 4\,\theta}{4}\right)_0^{\frac{\pi}{2}} = \frac{\pi a^2}{8}.$$

The entire area of the four loops $=\frac{\pi a^2}{2}$,
which is half the area of the circumscribed circle.

2. Find the entire area of the circle (Art. 135) $r=a\sin\theta$.
Ans. $\frac{\pi a^2}{4}$.

In the two following curves find the area described by the radius vector in moving from $\theta=0$ to $\theta=\frac{\pi}{4}$.

3. $r=\sec\theta+\tan\theta$. *Ans.* $\sqrt{2}-\frac{\pi}{8}$.

4. $r=a(1-\tan^2\theta)$. *Ans.* $\left(\frac{\pi}{2}-\frac{4}{3}\right)a^2$.

5. Find the entire area of the cardioid (Art. 141) $r=a(1-\cos\theta)$.
Ans. $\frac{3\pi a^2}{2}$, or six times the area of the generating circle.
Also find the area from $\theta=\frac{\pi}{4}$ to $\theta=\frac{3\pi}{4}$. *Ans.* $(3\pi-2)\frac{a^2}{8}$.

6. Find the area described by the radius vector in the parabola (Art. 139) $r=a\sec^2\frac{\theta}{2}$, from $\theta=0$ to $\theta=\frac{\pi}{2}$. *Ans.* $\frac{4a^2}{3}$.
Also find the area from $\theta=\frac{\pi}{3}$ to $\theta=\frac{2\pi}{3}$. *Ans.* $\frac{44a^2}{9\sqrt{3}}$.

7. Find the entire area of the lemniscate (Art. 143) $r^2=a^2\cos 2\theta$.
Ans. a^2.

8. Show that the area bounded by any two radii vectores of the reciprocal spiral (Art. 137) $r\theta=a$ is proportional to the difference between the lengths of these radii.

9. In the spiral of Archimedes (Art. 136), $r=a\theta$, find the area described by the radius vector in one entire revolution from $\theta=0$.
Ans. $\frac{4\pi^3a^2}{3}$.
Also find the area of the strip added by the nth revolution.
Ans. $8(n-1)\pi^3a^2$.

10. Find the area of the part of the circle (Art. 135)

$$r = a\sin\theta + b\cos\theta, \quad \text{from } \theta = 0 \text{ to } \theta = \frac{\pi}{2}.$$

Ans. $\dfrac{\pi(a^2+b^2)}{8} + \dfrac{ab}{2}.$

11. Find the area common to the two circles (Art. 135).

$$r = a\sin\theta + b\cos\theta, \quad r = a\cos\theta + b\sin\theta.$$

Ans. $\left(\dfrac{\pi}{2} + 2\tan^{-1}\dfrac{b}{a}\right)\dfrac{a^2+b^2}{4} - \dfrac{a^2-b^2}{4}$, where $a > b$.

12. Find the area of the loop of the Folium of Descartes (Art. 127)

$$r = \frac{3a\tan\theta\sec\theta}{1+\tan^3\theta}.$$

Ans. $\dfrac{3a^2}{2}.$

13. Show that the line $r = \dfrac{2a\sec\theta}{1+\tan\theta}$, $(x + y = 2a)$, divides the area of the loop of the preceding example in the ratio 2 : 1.

14. Find the entire area within the curve (Art. 145) $r = a\sin^3\dfrac{\theta}{3}$, no part being counted twice.

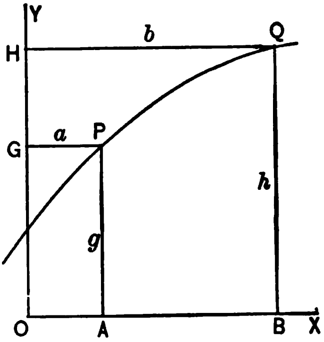
Ans. $(10\pi + 9\sqrt{3})\dfrac{a^2}{32}.$

249. Lengths of Curves. Rectangular Coördinates. To find the Length of the Arc ***PQ*** **between Two Given Points** ***P*** **and** ***Q*****.**

Let $OA = a$, $OB = b$.

Denoting the required length of arc by s, we have from (1), Art. 155,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$



Hence

$$s = \int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx + C,$$

and between the given limits

$$s = \int_a^b\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx, \quad \cdot \quad \cdot \quad (1)$$

the limits being the initial and final values of x.

We may also use the formula

$$s=\int_g^h\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy, \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (2)$$

the limits being the initial and final values of y,

$$g=OG, \text{ and } h=OH.$$

EXAMPLES

1. Find the length of the arc of the parabola $y^2=4ax$, from the vertex to the extremity of the latus rectum.

Here $$\frac{dy}{dx}=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}},$$

therefore $$s=\int_0^a\left(1+\frac{a}{x}\right)^{\frac{1}{2}}dx=\int_0^a\left(\frac{a+x}{x}\right)^{\frac{1}{2}}dx.$$

This may be integrated by Ex. 13, p. 305, making $b=0$.

$$\int\left(\frac{a+x}{x}\right)^{\frac{1}{2}}dx=\sqrt{ax+x^2}+a\log\left(\sqrt{a+x}+\sqrt{x}\right)$$

$$\int_0^a\left(\frac{a+x}{x}\right)^{\frac{1}{2}}dx=a\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]=2.29558\,a.$$

Or we may use the formula (2),

$$s=\int_0^{2a}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

$$x=\frac{y^2}{4a},\quad \frac{dx}{dy}=\frac{y}{2a}.$$

$$s=\int_0^{2a}\sqrt{1+\frac{y^2}{4a^2}}\,dy=\frac{1}{2a}\int_0^{2a}\sqrt{y^2+4a^2}\,dy$$

$$=\frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2}+\frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right]_0^{2a}$$

$$=a\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$$

2. Find the length of the arc of the semicubical parabola (Art. 130) $ay^2 = x^3$, from $x = \frac{a}{4}$ to $x = 5a$. *Ans.* $\frac{97\,a}{8}$.

3. Find the entire length of the arc of the hypocycloid (Art. 132)

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$ *Ans.* $6\,a$.

4. Find the length of the arc of the catenary (Art. 128)

$$y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}),$$

from $x = 0$ to the point (x, y). *Ans.* $\frac{a}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})$.

5. Find the length of the arc of the curve

$$y = \log \sec x, \text{ from } x = 0 \text{ to } x = \frac{\pi}{3}.$$

Ans. $\log(2 + \sqrt{3})$.

6. Find the length of the curve

$$6\,xy = x^4 + 3, \text{ from } x = 1 \text{ to } x = 2.$$ *Ans.* $\frac{17}{12}$.

7. Find the perimeter of the loop of the curve

$$9\,ay^2 = (x - 2\,a)(x - 5\,a)^2.$$ *Ans.* $4\sqrt{3}\,a$.

8. Find the length of that part of the evolute of the parabola (Art. 167) $27\,ay^2 = 4\,(x - 2\,a)^3$ included within the parabola $y^2 = 4\,ax$.

Ans. $4\,(3\sqrt{3} - 1)\,a$.

9. Find the length of the curve

$$y = \log \frac{e^x - 1}{e^x + 1}, \text{ from } x = 1 \text{ to } x = 2.$$

Ans. $\log(e + e^{-1})$.

10. Find the length of one quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

Ans. $\frac{a^2 + ab + b^2}{a + b}$.

11. The parametric equations of a curve are $x = e^{\theta} \sin \theta$, $y = e^{\theta} \cos \theta$. Find the length of arc from $\theta = 0$ to $\theta = \frac{\pi}{2}$. *Ans.* $\sqrt{2}\,(e^{\frac{\pi}{2}} - 1)$.

12. The parametric equations of the epicycloid, the radius of the fixed circle being a, and that of the rolling circle $\frac{a}{2}$, are (Art. 131)

$$x=\frac{a}{2}(3\cos\phi-\cos 3\phi),$$

$$y=\frac{a}{2}(3\sin\phi-\sin 3\phi),$$

ϕ being the angle of the fixed circle, over which the small circle has rolled.

Find the entire length of the curve. *Ans.* $12a$.

250. Lengths of Curves. Polar Coördinates. To find the Length of the Arc PQ between Two Given Points P and Q.

Let $POX=\alpha$, $QOX=\beta$.

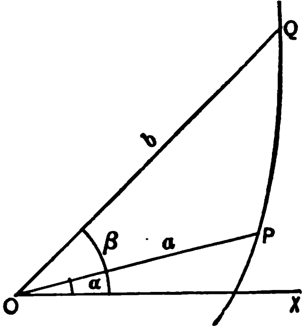
We have from (3), Art. 156,

$$ds=\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}}d\theta;$$

therefore

$$s=\int_\alpha^\beta\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}}d\theta, \quad . \quad . \quad (1)$$

the limits being the limiting values of θ.



Or we have $ds=\left[1+r^2\left(\frac{d\theta}{dr}\right)^2\right]^{\frac{1}{2}}dr$; (2), Art. 156,

therefore $$s=\int_a^b\left[1+r^2\left(\frac{d\theta}{dr}\right)^2\right]^{\frac{1}{2}}dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the limits being the limiting values of r. That is, $OP=a$, $OQ=b$.

EXAMPLES

1. Find the length of the arc of the spiral of Archimedes (Art. 136), $r=a\theta$, from the origin to the end of the first revolution.

Here $\frac{dr}{d\theta}=a$, and we have by (1),

$$s=\int_0^{2\pi}(a^2\theta^2+a^2)^{\frac{1}{2}}d\theta=a\int_0^{2\pi}(1+\theta^2)^{\frac{1}{2}}d\theta$$

$$=a\left[\frac{\theta\sqrt{1+\theta^2}}{2}+\frac{1}{2}\log(\theta+\sqrt{1+\theta^2})\right]_0^{2\pi}$$

$$=a\left[\pi\sqrt{1+4\pi^2}+\frac{1}{2}\log(2\pi+\sqrt{1+4\pi^2})\right].$$

Or we may use the formula (2)

$$s=\int_0^{2\pi a}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr.$$

$$\theta=\frac{r}{a},\qquad \frac{d\theta}{dr}=\frac{1}{a}.$$

$$s=\int_0^{2\pi a}\sqrt{1+r^2\frac{1}{a^2}}\,dr=\frac{1}{a}\int_0^{2\pi a}\sqrt{r^2+a^2}\,dr$$

$$=\frac{1}{a}\left[\frac{r}{2}\sqrt{r^2+a^2}+\frac{a^2}{2}\log(r+\sqrt{r^2+a^2})\right]_0^{2\pi a}$$

$$=a\left[\pi\sqrt{4\pi^2+1}+\frac{1}{2}\log(2\pi+\sqrt{4\pi^2+1})\right].$$

2. Find the entire length of the circle (Art. 135) $r=2a\sin\theta$.

Ans. $2\pi a$.

3. Find the length of the arc of the circle (Art. 135)

$r=a\sin\theta+b\cos\theta$, from $\theta=0$ to (r,θ). *Ans.* $\sqrt{a^2+b^2}\,\theta$.

4. Find the length of the logarithmic spiral (Art. 138) $r=e^{a\theta}$, from the point (r_1,θ_1) to (r_2,θ_2), using the formula (2), and the equation

$$\theta=\frac{\log r}{a}.\qquad \textit{Ans.}\ \frac{\sqrt{a^2+1}}{a}(r_2-r_1).$$

5. Find the entire length of the cardioid (Art. 141)

$$r = a(1 - \cos\theta). \qquad \textit{Ans. } 8a.$$

Also show that the arc of the upper half of the curve is bisected by

$$\theta = \frac{2\pi}{3}.$$

6. Solve Ex. 5 by using formula (2) and the equation $\theta = \text{vers}^{-1}\frac{r}{a}$.

7. Find the arc of the reciprocal spiral (Art. 137) $r\theta = a$, from

$$\theta = \frac{5}{12} \text{ to } \theta = \frac{3}{4}. \qquad \textit{Ans. } \left(\frac{14}{15} + \log\frac{4}{3}\right)a$$

8. Find the arc of the parabola (Art. 139) $r = a\sec^2\frac{\theta}{2}$ from

$$\theta = 0 \text{ to } \theta = \frac{\pi}{2}. \qquad \textit{Ans. } \left(\sec\frac{\pi}{4} + \log\tan\frac{3\pi}{8}\right)a$$

9. Find the entire length of the arc of the curve (Art. 145)

$$r = a\sin^3\frac{\theta}{3}. \qquad \textit{Ans. } \frac{3\pi a}{2}$$

Also show that the arc *AB* is one third of *OABC*.

Hence *OA*, *AB*, *BC*, are in arithmetical progression.

10. Find the entire length of the curve $r = a\sin^n\frac{\theta}{n}$, n being a positive integer.

See for integration Exs. 24, 25, p. 314.

$$\textit{Ans. } \frac{2\cdot4\cdot6\cdots n}{1\cdot3\cdot5\cdots(n-1)}2a, \text{ when } n \text{ is even.}$$

$$\frac{1\cdot3\cdot5\cdots n}{2\cdot4\cdot6\cdots(n-1)}\pi a, \text{ when } n \text{ is odd.}$$

251. Volumes of Surfaces of Revolution. To find the Volume generated by revolving about OX the Plane Area $APQB$.

Let $OA = a$, $OB = b$.

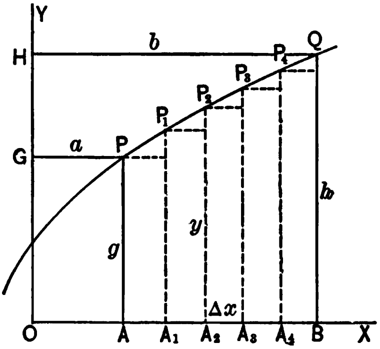
Let x and y be the coördinates of any point P_2 of the given curve.

It is evident that the rectangle $P_2A_2A_3$ will generate a right cylinder, whose volume is

$$\pi y^2\,\Delta x.$$

The sum of all these cylinders may be represented by

$$\pi\sum_a^b y^2\,\Delta x.$$



The required volume is the limit of the sum of the cylinders, as Δx approaches zero. That is,

$$V_x = \pi\int_a^b y^2\,dx.$$

Similarly the volume generated by revolving $PGHQ$ about OY is

$$V_y = \pi\int_g^h x^2\,dy,$$

where $OG = g$, and $OH = h$.

EXAMPLES

1. Find the volume generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

about its major axis, OX. This is called the *prolate spheroid.*

$$\frac{V}{2} = \pi\int_0^a y^2\,dx = \pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx = \frac{\pi b^2}{a^2}\left(a^2x - \frac{x^3}{3}\right)_0^a = \frac{2\,\pi ab^2}{3}.$$

$$V = \frac{4}{3}\pi ab^2.$$

2. Find the volume generated by revolving the ellipse about its minor axis, *OY*. This is called the *oblate spheroid.*

$$\frac{V}{2}=\pi\int_0^b x^2\,dy=\frac{\pi a^2}{b^2}\int_0^b (b^2-y^2)\,dy=\frac{2\pi a^2 b}{3}.$$

$$V=\frac{4}{3}\pi a^2 b.$$

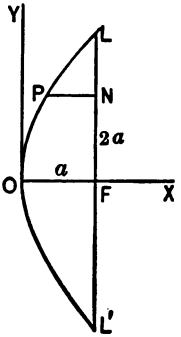
3. If the parabola $y^2=4ax$ is revolved about OX, show that the volume from $x=0$ to $x=2a$ is one third the volume from $x=2a$ to $x=4a$.

4. Find the volume generated by revolving the segment LOL^1 of the parabola about the latus rectum LL′.

Here $\frac{V}{2}=\pi\int_0^{2a}(PN)^2\,dy=\pi\int_0^{2a}(a-x)^2\,dy$

$$=\pi\int_0^{2a}\left(a-\frac{y^2}{4a}\right)^2 dy=\frac{16}{15}\pi a^3.$$

Ans. $\frac{32}{15}\pi a^3.$



5. Find the volume generated by revolving about OX one loop of the curve (Art. 134)

$a^4y^2=a^2x^4-x^6.$ *Ans.* $\frac{2}{35}\pi a^3.$

6. Find the entire volume generated by revolving about OX the hypocycloid (Art. 132) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$ *Ans.* $\frac{32}{105}\pi a^3.$

7. Find the volumes generated by revolving about OX, and about OY, the curve (Art. 133) $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

Ans. $V_x=\frac{32}{35}\pi ab^2.$ $V_y=\frac{4}{5}\pi a^2 b.$

8. The part of the line $\frac{x}{a}+\frac{y}{b}=1$, intercepted between the coördinate axes, is revolved about the line $x=2a$. Find the included volume. *Ans.* $\frac{7}{3}\pi a^2b$.

9. The segment of the parabola, $x^2-3x+2y=0$, above OX, is revolved about OX. Find the volume generated. *Ans.* $\frac{81\pi}{40}$.

10. A segment of a circle is revolved about a diameter parallel to its chord. Show that the volume generated is equal to that of a sphere whose diameter is equal to the chord.

11. Find the volume generated by revolving about OY the witch (Art. 126), $y=\frac{8a^3}{x^2+4a^2}$, from $(0, 2a)$ to $y=a$. *Ans.* $4(\log 4-1)\pi a^3$.

12. Find the volume generated by revolving the upper half, $ABA'OA$, of the curve (Art. 133) $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$, about the tangent at B.

Ans. $\left(\frac{3\pi}{4}-\frac{32}{35}\right)\pi ab^2$.

13. Find the volume generated by revolving about OX the area included between the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the parabola $2ay^2=3b^2x$.

Ans. $\frac{19}{48}\pi ab^2$.

14. A segment of the catenary (Art. 128), $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$, by a chord through the points $x=\pm a\log 2$, is revolved about the tangent at the vertex. Find the volume generated.

Ans. $\left(\frac{33}{16}-\frac{23}{8}\log 2\right)\pi a^3$.

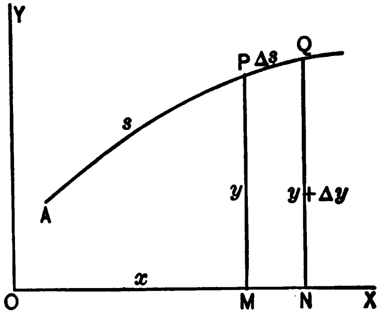
15. Find the volume generated by revolving about the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the segment cut off by the latus rectum. *Ans.* $2\pi\left(ab^2-\frac{b^4}{3a}-ab\sqrt{a^2-b^2}\sin^{-1}\frac{b}{a}\right)$

252. Derivative of Area of Surface of Revolution. In order to obtain the formula for the surface generated by the revolution of a given arc, it is necessary to find the derivative of this surface with respect to the arc.

Let S denote the surface generated by revolving about OX the arc s, AP.

Using for abbreviation the expression "Surf ( )" to denote "the surface generated by revolving ( ) about OX," we have

$$S = \text{Surf}(s), \quad \Delta S = \text{Surf}(\Delta s).$$



This may be written

$$\Delta S = \frac{\text{Surf}(\Delta s)}{\text{Surf}(\text{Chord } PQ)} \text{Surf}(\text{Chord } PQ). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the surface generated by the chord PQ is the convex surface of the frustum of a right cone, which is the product of the slant height by the circumference of a section midway between the bases.

Hence
$$\text{Surf}(\text{Chord } PQ) = 2\pi\left(\frac{PM + QN}{2}\right)\text{Chord } PQ$$
$$= 2\pi\frac{y + y + \Delta y}{2}\text{Chord } PQ$$
$$= \pi(2y + \Delta y)\text{Chord } PQ.$$

Substituting this for the last factor in (1), and dividing both sides by Δs, we have

$$\frac{\Delta S}{\Delta s} = \frac{\text{Surf}(\Delta s)}{\text{Surf}(\text{Chord } PQ)}\pi(2y + \Delta y)\frac{\text{Chord } PQ}{\Delta s}.$$

Taking the limit of each member, as Δs approaches zero, noticing that

$$\text{Lim}_{\Delta s = 0}\frac{\text{Surf}(\Delta s)}{\text{Surf}(\text{Chord } PQ)} = 1,$$

and $$\text{Lim}_{\Delta s=0}\frac{\text{Chord } PQ}{\Delta s}=1, \text{ we have}$$

$$\frac{dS}{ds}=\text{Lim}_{\Delta s=0}\frac{\Delta S}{\Delta s}=2\pi y.$$

Similarly if OY is the axis of revolution,

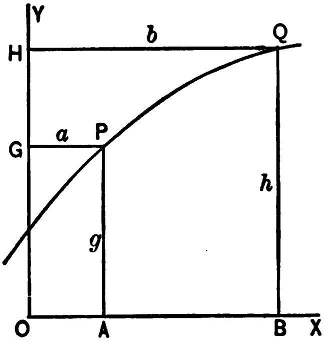
$$\frac{dS}{ds}=2\pi x.$$

253. Areas of Surfaces of Revolution. To find the Area of the Surface generated by revolving about OX the Arc PQ.

By the preceding article we have

$$\frac{dS}{ds}=2\pi y;$$

hence $$S=\int 2\pi y\,ds.$$



To express this in terms of x and y, we have from (1), Art. 155,

$$ds=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx,$$

which gives

$$S_x=2\pi\int_a^b y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx. \quad . \quad (1)$$

If OY is the axis of revolution,

$$S_y=2\pi\int x\,ds=2\pi\int_a^b x\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx. \quad . \quad . \quad . \quad (2)$$

Or we may use $$ds=\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}dy,$$

and instead of (1) we have

$$S_x=2\pi\int_g^h y\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}dy. \quad . \quad . \quad . \quad (1')$$

and instead of (2) $$S_y=2\pi\int_g^h x\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}dy. \quad . \quad . \quad . \quad (2')$$

EXAMPLES

1. Find the area of the surface generated by revolving about OX the hypocycloid (Art. 132) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Here $$y=(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}},\qquad \frac{dy}{dx}=-(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{1}{2}}x^{-\frac{1}{3}}.$$

Using (1) $$\frac{1}{2}S_x=2\pi\int_0^a(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}\left[1+\frac{a^{\frac{2}{3}}-x^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right]^{\frac{1}{2}}dx$$

$$=2\pi\int_0^a(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}\,dx=2\pi a^{\frac{1}{3}}\int_0^a(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}x^{-\frac{1}{3}}dx$$

$$=\frac{6\pi a^2}{5}.\qquad\qquad S_x=\frac{12\pi a^2}{5}.$$

Or we may use (1′) $$\frac{dx}{dy}=-(a^{\frac{2}{3}}-y^{\frac{2}{3}})^{\frac{1}{2}}y^{-\frac{1}{3}}.$$

$$\frac{1}{2}S_x=2\pi\int_0^a y\left[1+\frac{a^{\frac{2}{3}}-y^{\frac{2}{3}}}{y^{\frac{2}{3}}}\right]^{\frac{1}{2}}dy=2\pi a^{\frac{1}{3}}\int_0^a y^{\frac{2}{3}}\,dy=\frac{6\pi a^2}{5}.$$

2. Show that the area of the surface generated by revolving the parabola $y^2=4ax$, about OX, from $x=0$ to $x=3a$, is one eighth of that from $x=3a$ to $x=15a$.

3. Find the area of the surface generated by revolving about OX the loop of the curve $9ay^2=x(3a-x)^2$. *Ans.* $3\pi a^2$.

4. Find the surface generated by revolving about OX, the arc of the curve $6a^2xy=x^4+3a^4$, from $x=a$ to $x=2a$. *Ans.* $\frac{47}{16}\pi a^2$.

5. The arc of the preceding curve from $x=a$ to $x=3a$, revolves about OY. What is the surface generated? *Ans.* $(20+\log 3)\pi a^2$.

6. Find the surface generated by revolving about OY the curve $4y=x^2-2\log x$, from $x=1$ to $x=4$. *Ans.* 24π.

7. Find the entire surface generated by revolving about OX the ellipse $3x^2 + 4y^2 = 3a^2$. *Ans.* $\left(\frac{3}{2} + \frac{\pi}{\sqrt{3}}\right)\pi a^2$.

8. Find the entire surface generated by revolving about OY the preceding ellipse. *Ans.* $(4 + 3\log 3)\frac{\pi a^2}{2}$.

9. Find the surface generated by revolving about OX a loop of the curve $8a^2y^2 = a^2x^2 - x^4$. *Ans.* $\frac{\pi a^2}{4}$.

10. An arc, subtending an angle 2α, of a circle whose radius is a, revolves about its chord. Find the surface generated.

Ans. $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$.

11. The arc of the catenary (Art. 128) $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, from $x = a$ to $x = 2a$, revolves about OY. Find the surface generated.

Ans. $(e^2 + 2e^{-1} - 3e^{-2})\pi a^2$.

12. The parametric equations of a curve are

$$x = e^{\theta}\sin\theta, \quad y = e^{\theta}\cos\theta.$$

Find the surface generated by revolving the arc from $\theta = 0$ to $\theta = \frac{\pi}{2}$, about OX. *Ans.* $\frac{2\sqrt{2}\,\pi}{5}(e^{\pi} - 2)$.

13. Find the surface generated by revolving about OY the arc of the preceding example. *Ans.* $\frac{2\sqrt{2}\,\pi}{5}(2e^{\pi} + 1)$.

14. The parametric equations of the epicycloid, the radius of the fixed circle being a, and that of the rolling circle $\frac{a}{2}$ (Art. 131)

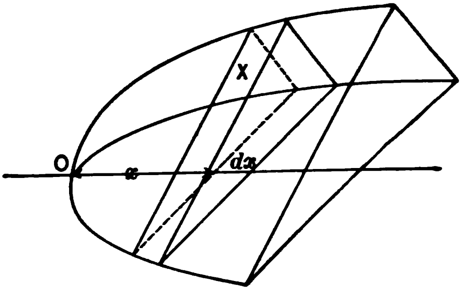
are $x = \frac{3a}{2}\cos\phi - \frac{a}{2}\cos 3\phi, \quad y = \frac{3a}{2}\sin\phi - \frac{a}{2}\sin 3\phi.$

Find the entire surface generated by revolving the curve about OX.

Ans. $\frac{9}{2}\pi^2a^2$.

15. Find the surface generated by revolving one arch of the preceding curve about OY. *Ans.* $6\pi a^2$.

254. Volume by Area of Section. The volume of a solid may be found by a single integration, when the areas of parallel sections can be expressed in terms of their perpendicular distances from a fixed point.



Let us denote this distance by x, and the area of a section, supposed to be a function of x, by X.

The volume included between two sections separated by the distance dx will ultimately be $X\,dx$, and we have for the volume of the solid

$$V=\int X\,dx,$$

the limits being the initial, and final, values of x.

EXAMPLES

1. Find the volume of a pyramid or cone having any base.

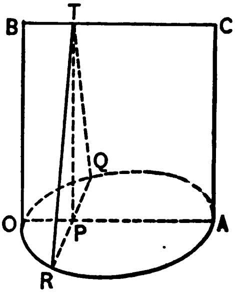
Let A be the area of the base, and h the altitude.

Let x denote the perpendicular distance from the vertex of a section parallel to the base. Calling the area of this section X, we have, by solid geometry,

$$\frac{X}{A}=\frac{x^2}{h^2},\qquad X=\frac{Ax^2}{h^2}.$$

Hence,

$$V=\int_0^h X\,dx=\frac{A}{h^2}\int_0^h x^2\,dx=\frac{Ah^3}{h^2 3}=\frac{Ah}{3}.$$



2. Find the volume of a right conoid with circular base, the radius of base being a, and altitude h.

$$OA=BC=2\,a,\quad BO=CA=h.$$

The section RTQ, perpendicular to OA, is an isosceles triangle

Let $x = OP$; then

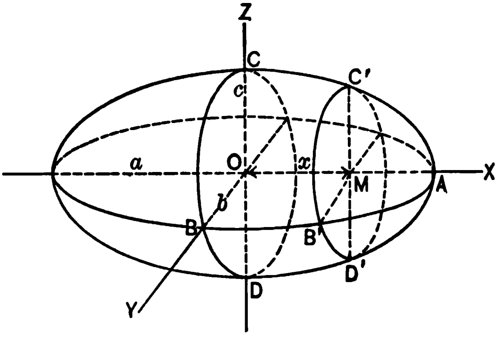
$$X = \text{area } RTQ = PT \times PQ = h\sqrt{2ax - x^2}.$$

Hence, $$V = \int_0^{2a} X\,dx = h\int_0^{2a} \sqrt{2ax - x^2}\,dx = \frac{\pi a^2 h}{2}.$$

This is one half the cylinder of the same base and altitude.

3. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \text{. . . . .} \quad (1)$$



Let us find the area of a section $C'B'D'$ perpendicular to OX, at the distance from the origin $OM = x$.

This section is an ellipse whose semi-axes are MB' and MC'.

To find MB', let $z = 0$ in (1), and we have

$$y = MB' = \frac{b}{a}\sqrt{a^2 - x^2}.$$

To find MC', let $y = 0$ in (1), and we have

$$z = MC' = \frac{c}{a}\sqrt{a^2 - x^2}.$$

The area of the ellipse (Ex. 2, p. 321) is $\pi (MB')(MC')$.

Hence, $$X = \frac{\pi bc}{a^2}(a^2 - x^2),$$

and $$V = 2\int_0^{a} X\,dx = \frac{2\pi bc}{a^2}\int_0^{a} (a^2 - x^2)\,dx = \frac{4}{3}\pi abc.$$

4. A rectangle moves from a fixed point, one side varying as the distance from the point, and the other as the square of this distance. At the distance of 20 feet the rectangle becomes a square of 3 feet. What is the volume then generated? *Ans.* 45 cubic feet.

5. The axes of two right circular cylinders having equal bases, radius a, intersect at right angles. Find the volume common to the two. (See Fig., p. 359.) *Ans.* $\frac{16a^3}{3}$.

6. A torus is generated by a circle, radius b, revolving about an axis in its plane, a being the distance of the centre of the circle from the axis. Find the volume by means of sections perpendicular to the axis. *Ans.* $2\pi^2ab^2$.

7. A football is 16 inches long, and a plane section containing a seam of the cover is an ellipse 8 inches broad. Find the volume of the ball, assuming that the leather is so stiff that every plane cross-section is a square. *Ans.* $341\frac{1}{3}$ cu. in.

8. Given a right cylinder, altitude h, and radius of base a. Through a diameter of the upper base two planes are passed, touching the lower base on opposite sides. Find the volume included between the planes. *Ans.* $\left(\pi - \frac{4}{3}\right)a^2h$.

9. Two cylinders of equal altitude h have a circle of radius a, for their common upper base. Their lower bases are tangent to each other. Find the volume common to the two cylinders.

Ans. $\frac{4a^2h}{3}$.

CHAPTER XXIX

SUCCESSIVE INTEGRATION

255. Definite Double Integral. — A double integral is the integral of an integral.

Thus, x and y being independent variables, the definite double integral,

$$\int_b^a \int_d^c f(x, y)dx\,dy,$$

indicates the following operations:

Treating x as a constant, integrate $f(x, y)$ with respect to y between the limits d and c; then integrate the result with respect to x between the limits b and a.*

For example,

$$\int_a^{2a}\int_0^b x^2(b-y)dx\,dy = \int_a^{2a} x^2\left(by - \frac{y^2}{2}\right)_0^b dx = \int_a^{2a} x^2\frac{b^2}{2}\,dx$$

$$= \frac{b^2}{2}\frac{x^3}{3}\bigg|_a^{2a} = \frac{7\,a^3b^2}{6}.$$

Notice that the order of the two integrations is indicated in the given definite integral by the order of the differentials $dx\,dy$, taken from right to left, the pairs of limits $\int_a^{2a}\int_0^b$ being used in the same order.

It should be said, however, that the order of the integrations is denoted differently by different writers.

256. Variable Limits. — The limits of the first integration, instead of being constants, are often functions of the variable of the second integration.

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[*] Using parentheses, this might be represented by $\int_b^a\left(\int_d^c f(x, y)dy\right)dx.$

For example,

$$\int_0^a \int_{y-a}^{2y} xy\,dy\,dx = \int_0^a \left(\frac{x^2}{2}\right)_{y-a}^{2y} y\,dy = \frac{1}{2}\int_0^a (3\,y^3 + 2\,ay^2 - a^2y)dy = \frac{11\,a^4}{24}.$$

As another example,

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} (x+y)dx\,dy = \int_0^a \left(xy + \frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right)dx = \frac{2\,a^3}{3}.$$

When the limits are all constants, as in Art. 255, the order of the integrations may be reversed without affecting the result. That is,

$$\int_a^{2a} \int_0^b x^2(b-y)dx\,dy = \int_0^b \int_a^{2a} x^2(b-y)dy\,dx.$$

Where the definite integral has variable limits, the order of integrations can be changed only by new limits adapted to the new order.

257. Triple Integrals. — A similar notation is used for three successive integrations. Thus

$$\int_b^a \int_0^b \int_a^{2a} x^2y^2z\,dx\,dy\,dz = \int_b^a \int_0^b \frac{3\,a^2}{2}x^2y^2dx\,dy$$

$$= \frac{3\,a^2}{2}\int_b^a \frac{b^3}{3}x^2\,dx = \frac{a^2b^3}{2}\left(\frac{a^3}{3} - \frac{b^3}{3}\right) = \frac{a^2b^3}{6}(a^3 - b^3).$$

EXAMPLES

Evaluate the following definite integrals:

1. $\int_0^a \int_0^b xy\,(x-y)\,dx\,dy = \frac{a^2b^2}{6}(a-b).$

2. $\int_b^a \int_\beta^\alpha r^2 \sin\theta\,dr\,d\theta = \frac{a^3-b^3}{3}(\cos\beta - \cos\alpha).$

3. $$\int_a^{2a}\int_y^{\frac{y^2}{a}}(x+y)\,dy\,dx=\frac{67\,a^3}{20}.$$

4. $$\int_{\frac{b}{2}}^{b}\int_0^{\frac{r}{b}}r\,dr\,d\theta=\frac{7\,b^2}{24}.$$

5. $$\int_0^{\pi}\int_0^{a(1+\cos\theta)}r^2\sin\theta\,d\theta\,dr=\frac{4\,a^3}{3}.$$

6. $$\int_0^a\int_y^{10\,y}\sqrt{xy-y^2}\,dy\,dx=6\,a^3.$$

7. $$\int_0^1\int_0^{v^2}e^{\frac{w}{v}}dv\,dw=\frac{1}{2}.$$

8. $$\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{a}r^4\,d\theta\,dr=\left(\pi-\frac{16}{15}\right)\frac{a^5}{10}.$$

9. $$\int_0^{\frac{\pi}{6}}\int_{\theta}^{2\theta}\sin(2\,\phi-\theta)\,d\theta\,d\phi=\frac{1}{12}.$$

10. $$\int_0^{\pi}\int_0^{t}t\sin^2\phi\,dt\,d\phi=\frac{\pi^3}{6}+\frac{\pi}{8}.$$

11. $$\int_0^a\int_{\frac{x^2}{a}}^{x}\frac{x\,dx\,dy}{x^2+y^2}=\frac{a}{2}\log 2.$$

12. $$\int_0^a\int_0^b\int_0^c(x^2+y^2+z^2)\,dx\,dy\,dz=\frac{abc}{3}(a^2+b^2+c^2).$$

13. $$\int_1^e\int_1^x\int_0^{\frac{\pi}{2\,xy}}\sin(xyz)\,dx\,dy\,dz=\frac{1}{2}.$$

14. $$\int_0^a\int_0^{2u}\int_{u-v}^{v}uvw\,du\,dv\,dw=\frac{5\,a^6}{18}.$$

15. $$\int_0^{\log 2}\int_0^x\int_0^{x+y}e^{x+y+z}\,dx\,dy\,dz=\frac{5}{8}.$$

CHAPTER XXX

APPLICATIONS OF DOUBLE INTEGRATION

258. Moment of Inertia. If $r_1, r_2, r_3, \cdots, r_n$ are the distances from a given line, of n particles of masses $m_1, m_2, m_3, \cdots, m_n$, the sum

$$m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots + m_nr_n^2 = \sum(mr^2)$$

is defined in treatises on mechanics as the *moment of inertia* of the system about the given line.

The moment of inertia of a continuous solid about a given line is the sum of the products obtained by multiplying the mass of each infinitesimal portion of the solid by the square of its distance from the given line.

The summation is then effected by integration, and we have for the moment of inertia of a body of mass M,

$$I = \int r^2\,dM.$$

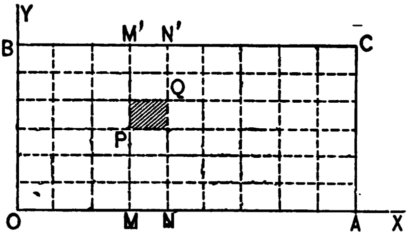
259. Moment of Inertia of a Plane Area. The moment of inertia of a given plane area about a given point O may be defined as the sum of the products obtained, by multiplying the area of each infinitesimal portion by the square of its distance from O.

This may be regarded as the moment of inertia of a thin plane sheet of uniform thickness and density, about a line through O perpendicular to the plane, the mass of a square unit of the sheet being taken as unity.

We shall illustrate double integration by finding the moment of inertia of certain areas.

260. Double Integration. Rectangular Coördinates. To find the moment of inertia of the rectangle $OACB$ about O.

Let $OA = a$, $OB = b$.



Suppose the rectangle divided into rectangular elements by lines parallel to the coördinate axes. Let x, y, which are to be regarded as independent variables, be the coördinates of any point of intersection as P, and $x + dx$, $y + dy$ the coördinates of Q. Then the area of the element PQ is $dx\,dy$.

Moment of inertia of $PQ = \overline{OP}^2 \cdot dx\,dy = (x^2 + y^2)\,dx\,dy$.

The moment of inertia of the entire rectangle $OACB$ is the sum of all the terms obtained from $(x^2 + y^2)\,dx\,dy$, by varying x from 0 to a, and y from 0 to b.

If we suppose x to be constant, while y varies from 0 to b, we shall have the terms that constitute a vertical strip $MNN'M'$.

Hence

$$\text{Moment of inertia of } MNN'M' = dx\int_0^b (x^2 + y^2)\,dy$$

$$= dx\left(x^2y + \frac{y^3}{3}\right)_0^b = \left(bx^2 + \frac{b^3}{3}\right)dx.$$

Having thus found the moment of a vertical strip, we may sum all these strips by supposing x in this result to vary from 0 to a. That is, the moment of inertia of $OACB$,

$$I = \int_0^a \left(bx^2 + \frac{b^3}{3}\right)dx = \frac{a^3b + ab^3}{3}.$$

The preceding operations are those represented by the double integral,

$$I = \int_0^a \int_0^b (x^2 + y^2)\,dx\,dy.$$

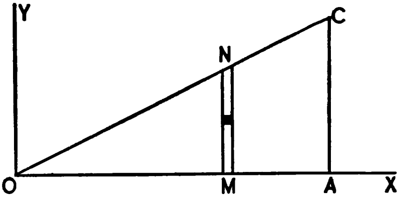
If we first collect all the elements in a *horizontal* strip, and then sum these horizontal strips, we have

$$I=\int_0^b\int_0^a(x^2+y^2)\,dy\,dx=\frac{a^3b+ab^3}{3}.$$

261. Variable Limits. To find the moment of inertia of the right triangle OAC about O.

Let $OA=a$, $AC=b$. The equation of OC is

$$y=\frac{b}{a}x.$$



This differs from the preceding problem only in the limits of the first integration. In collecting the elements in a vertical strip MN, y varies from 0 to MN. But MN is no longer a constant as in Art. 260, but varies with OM, according to the equation of OC, $y=\frac{b}{a}x$.

Hence the limits of y are 0 and $\frac{b}{a}x$.

In collecting all the vertical strips by the second integration, x varies from 0 to a, as in Art. 260.

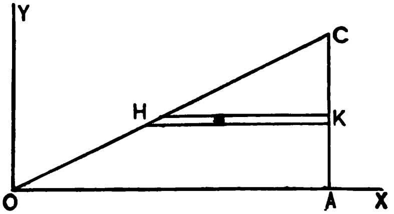
Thus we have for the moment of inertia of OAC,

$$I=\int_0^a\int_0^{\frac{bx}{a}}(x^2+y^2)\,dx\,dy=ab\left(\frac{a^2}{4}+\frac{b^2}{12}\right).$$

By supposing the triangle composed of *horizontal* strips as HK, we shall find

$$I=\int_0^b\int_{\frac{ay}{b}}^a(x^2+y^2)\,dy\,dx$$

$$=ab\left(\frac{a^2}{4}+\frac{b^2}{12}\right).$$



262. Plane Area as a Double Integral. If in Art. 260 we omit the factor (x^2+y^2), we shall have, instead of the moment of inertia, the area of the given surface.

That is,
$$\text{Area} = \int\int dx\,dy = \int\int dy\,dx,$$
the limits being determined as before.

EXAMPLES

1. Find the moment of inertia about the origin of the right triangle formed by the coördinate axes and the line joining the points $(a, 0)$, $(0, b)$.

$$\textit{Ans.}\quad \int_0^a\int_0^{\frac{b(a-x)}{a}}(x^2+y^2)\,dx\,dy = \frac{ab(a^2+b^2)}{12}.$$

2. Find the moment of inertia about the origin of the circle $x^2+y^2=a^2$.

$$\textit{Ans.}\quad 4\int_0^a\int_0^{\sqrt{a^2-x^2}}(x^2+y^2)\,dx\,dy = \frac{\pi a^4}{2}.$$

3. Find by a double integration the area between a straight line and a parabola, each of which joins the origin and the point (a, b), the axis of X being the axis of the parabola.

$$\textit{Ans.}\quad \int_0^a\int_{\frac{bx}{a}}^{b\sqrt{\frac{x}{a}}}dx\,dy = \int_0^b\int_{\frac{ay^2}{b^2}}^{\frac{ay}{b}}dy\,dx = \frac{ab}{6}.$$

4. Find the moment of inertia about the origin of the preceding area.

$$\textit{Ans.}\quad \frac{ab}{4}\left(\frac{a^2}{7}+\frac{b^2}{5}\right).$$

5. Find by a double integration the area included between the circle $x^2+y^2=10\,ay$, the line $3\,x+y=10\,a$, and the axis of Y.

$$\textit{Ans.}\quad \int^a\int_0^{\sqrt{10\,ay-y^2}}dy\,dx + \int_a^{10\,a}\int_0^{\frac{10\,a-y}{3}}dy\,dx$$

$$= \int_0^{3\,a}\int_{5\,a-\sqrt{25\,a^4-x^2}}^{10\,a-3\,x}dx\,dy = \frac{5\,a^2}{2}\left(3+5\sin^{-1}\frac{3}{5}\right).$$

6. Find the moment of inertia about the origin of the area between the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the line $\frac{x}{a}+\frac{y}{b}=1$.

Ans. $\left(\frac{\pi}{16}-\frac{1}{12}\right)(a^3b+ab^3)$.

7. Find the moment of inertia about the origin of the area between the parabola $y^2=4a(a-x)$, the circle $x^2+y^2=a^2$, and the axis of Y.

Ans. $\left(\frac{48}{35}-\frac{\pi}{8}\right)a^4$.

8. Find by a double integration the area included between the parabolas $y^2=3x$, and $y^2=12(60-x)$. *Ans.* 960.

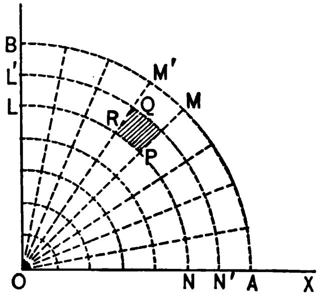
9. Find the moment of inertia of the area included between the parabola $y^2=4ax$, $x=4a$, and the axis of X, about the focus of the parabola.

Ans. $\frac{2336}{35}a^4$.

10. Find the moment of inertia of the area included between the lines $y=2x$, $x+2y=5a$, and the axis of X, about the intersection of the first two lines.

Ans. $\frac{125a^4}{6}$

263. Double Integration. Polar Coördinates. To find the area of the quadrant of a circle AOB, whose radius is a.



In rectangular coördinates, Art. 260, the lines of division consist of two systems, for one of which x is constant, and for the other, y is constant.

So in polar coördinates, we have one system of straight lines through the origin, for each of which θ is constant, and another system of circles about the origin as centre, for each of which r is constant.

Let r, θ, which are to be regarded as independent variables, be the coördinates of any point of intersection as P, and $r+dr$, $\theta+d\theta$, the coördinates of Q. Then the area of PQ is ultimately

$$PR \times RQ = r\,d\theta \cdot dr.$$

If we first integrate, regarding θ constant while r varies from 0 to a, we collect all the elements in any sector MOM'.

The second integration sums all the sectors, by varying θ from 0 to $\frac{\pi}{2}$.

Hence $$\text{Area } BOA = \int_0^{\frac{\pi}{2}}\int_0^a r\,d\theta\,dr = \frac{\pi a^2}{4}.$$

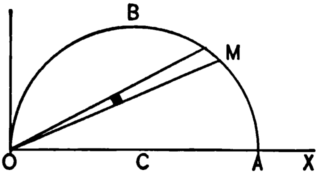
If we reverse the order of integration, integrating first with respect to θ, and afterwards with respect to r, we collect all the elements in a circular strip $NLL'N'$, and sum all these strips. This is written

$$\text{Area } BOA = \int_0^a\int_0^{\frac{\pi}{2}} r\,dr\,d\theta.$$

264. If the moment of inertia about O is required, we have for the moment of inertia of PQ, $r^2 \cdot r\,d\theta\,dr$. Hence, the moment of BOA is

$$I = \int_0^{\frac{\pi}{2}}\int_0^a r^3\,d\theta\,dr = \int_0^a\int_0^{\frac{\pi}{2}} r^3\,dr\,d\theta = \frac{\pi a^4}{8}.$$

265. Variable Limits. To find the moment of inertia about O of the semicircle OBA with radius $OC = a$, the origin being on the circumference.



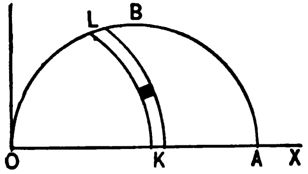
The polar equation of the circle is $r = 2a\cos\theta$. If we integrate first with respect to r, then with respect to θ, we shall have

$$I = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^3\,d\theta\,dr = \frac{3\pi a^4}{4}.$$

Here, in collecting the elements in a radial strip OM, r varies from 0 to OM. But OM varies with θ, according to the equation of the circle $r = 2a\cos\theta$. Hence the limits are 0 and $2a\cos\theta$.

In collecting all these radial strips for the second integration, θ varies from 0 to $\frac{\pi}{2}$.

By supposing the area composed of concentric circular strips about O, as LK, we find



$$I=\int_0^{2a}\int_0^{\cos^{-1}\left(\frac{r}{2a}\right)} r^3\,dr\,d\theta=\frac{3\pi a^4}{4}.$$

EXAMPLES

1. Find the moment of inertia about the origin of the area included between the two circles, $r=a\sin\theta$ and $r=b\sin\theta$, where $a>b$ (Art. 135).

Ans. $I=\int_0^{\pi}\int_{b\sin\theta}^{a\sin\theta} r^3\,d\theta\,dr=\frac{3}{32}\pi(a^4-b^4)$.

2. Find the moment of inertia about the origin of the area between the parabola (Art. 139), $r=a\sec^2\frac{\theta}{2}$, its latus rectum, and OX.

Ans. $\frac{48\,a^4}{35}$.

3. Find the moment of inertia about its centre of the area of the lemniscate (Art. 143) $r^2=a^2\cos 2\theta$.

Ans. $\frac{\pi a^4}{8}$.

4. Find by double integration the entire area of the cardioid (Art. 141) $r=a(1-\cos\theta)$.

Ans. $\frac{3\pi a^2}{2}$.

5. Find the moment of inertia about the origin of the area of the preceding cardioid.

Ans. $\frac{35\pi a^4}{16}$.

6. Find the moment of inertia about its centre, of the entire area of the four-leaved rose (Art. 144) $r = a \sin 2\theta$. *Ans.* $\frac{3\pi a^4}{16}$.

7. Find by a double integration the area of one loop of the lemniscate (Art. 143) outside the circle $2r^2 = a^2$. *Ans.* $\left(\sqrt{3} - \frac{\pi}{3}\right)\frac{a^2}{4}$.

8. Find the moment of inertia of the area of the preceding example about the centre of the lemniscate. *Ans.* $\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)\frac{a^4}{16}$.

266. Volumes and Surfaces of Revolution. Polar Coördinates. If in the figure of Art. 263 we suppose a revolution about OX, the volume generated by the infinitesimal area PQ is the product of this area by the circumference through which it revolves, that is, $2\pi r \sin\theta \cdot r\, d\theta\, dr$.

Hence for the entire volume

$$V = 2\pi \int\int r^2 \sin\theta\, d\theta\, dr,$$

the limits being determined as in Art. 263.

If the revolution is about OY,

$$V = 2\pi \int\int r^2 \cos\theta\, d\theta\, dr.$$

The area of the surface generated about OX is (Art. 253)

$$S = 2\pi \int y\, ds = 2\pi \int r \sin\theta \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta.$$

EXAMPLES

1. Find the volume generated by revolving the cardioid (Art. 141) $r = a(1 - \cos\theta)$ about OX. *Ans.* $\frac{8}{3}\pi a^3$, twice the inscribed sphere.

2. Show that the entire volume generated by revolving the four-leaved rose (Art. 144) $r = a \sin 2\theta$, about OX, is $\frac{16}{35}$ of the volume of the circumscribed sphere.

3. Find the volume generated by revolving one loop of the four-leaved rose $r = a \sin 2\theta$ about the axis of the loop.

Ans. $\left(8\sqrt{2} - 9\right)\dfrac{2\pi a^3}{105}$.

4. Find the volume generated by revolving the lemniscate (Art 143) $r^2 = a^2 \cos 2\theta$ about OY.

Ans. $\dfrac{\pi^2 a^3 \sqrt{2}}{8}$.

5. Find the volume generated by revolving the lemniscate about OX.

Ans. $\dfrac{\pi a^3}{2}\left[\dfrac{1}{\sqrt{2}} \log(\sqrt{2}+1) - \dfrac{1}{3}\right]$.

6. Find area of surface generated by revolving the cardioid $r = a(1 - \cos\theta)$ about OX.

Ans. $\dfrac{32\pi a^2}{5}$.

7. Find surface generated by revolving the entire lemniscate $r^2 = a^2 \cos 2\theta$ (Art. 143) about OX. *Ans.* $2\pi a^2(2 - \sqrt{2})$.

8. Solve preceding example revolving about OY. *Ans.* $2\pi a^2\sqrt{2}$.

9. Find surface generated by revolving the entire lemniscate $r^2 = a^2 \sin 2\theta$ (Art. 143) about the tangent OT'. *Ans.* $4\pi a^2$.

10. Find the moment of inertia of a sphere (radius a) about a diameter, m being the mass of a unit of volume.

Ans. $\dfrac{8\pi a^5 m}{15}$.

CHAPTER XXXI

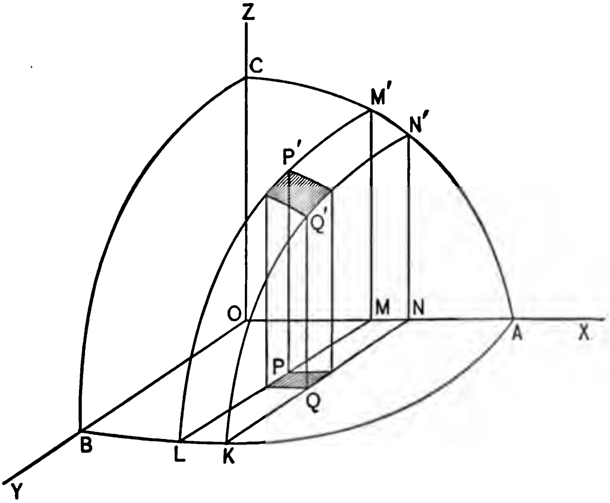
SURFACE, VOLUME, AND MOMENT OF INERTIA OF ANY SOLID

267. To find the Area of Any Surface, whose Equation is given between Three Rectangular Coördinates, x, y, z.

Let this equation be

$$z=f(x, y).$$

Suppose the given surface to be divided into elements by two series of planes, parallel respectively to XZ and YZ. These planes will also divide the plane XY into elementary rectangles, one of which is PQ, the projection upon the plane XY of the corresponding element of the surface $P'Q'$.



Let x, y, z be the coördinates of P' and $x+dx$, $y+dy$, $z+dz$, of Q'.

Since PQ is the projection of $P'Q'$, the area of PQ is equal to that of $P'Q'$ multiplied by the cosine of the inclination of $P'Q'$ to the plane XY. This angle is evidently that made by the tangent plane at P' with the plane XY. Denoting this angle by γ,

$$\text{Area } PQ = \text{Area } P'Q' \cdot \cos\gamma,$$

$$\text{Area } P'Q' = \text{Area } PQ \cdot \sec\gamma.$$

We see from the figure that

$$\text{Area } PQ = dx\,dy.$$

Also from (8), Art. 110, $\sec\gamma = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}}$,

where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are partial derivatives, taken from the equation of the given surface $z = f(x, y)$.

Hence $\quad \text{Area } P'Q' = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dx\,dy.$

If S denote the required surface,

$$S = \int\!\!\int \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dx\,dy, \quad . \quad . \quad . \quad (1)$$

the limits of the integration depending upon the projection, on the plane XY, of the surface required.

For example, suppose the surface ABC to be one eighth of the surface of a sphere whose equation is

$$x^2 + y^2 + z^2 = a^2.$$

Here $\quad \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}.$

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2 + y^2}{z^2} = \frac{a^2}{z^2} = \frac{a^2}{a^2 - x^2 - y^2}.$$

Substituting in (1), we have

$$S = a\int\int\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}.$$

This is to be integrated over the region *OBA*, the projection of the required surface on the plane *XY*.

The equation of the boundary *AB* is

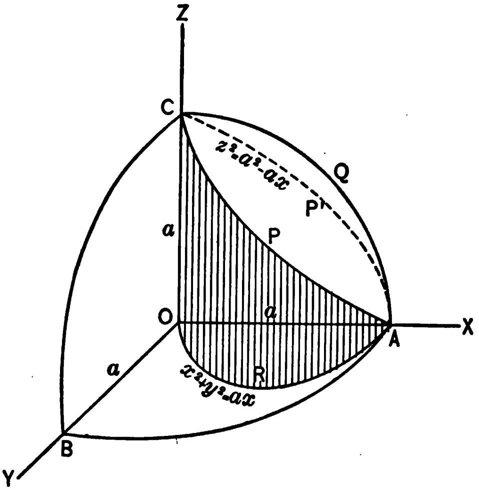
$$x^2+y^2=a^2.$$

Integrating first with respect to y, we collect all the elements in a strip *M'N'KL*, y varying from zero to *ML*, that is, between the limits 0 and $\sqrt{a^2-x^2}$.

Integrating afterwards with respect to x, we sum all the strips, to obtain the required surface *ABC*, x varying from 0 to a.

Hence $$S = a\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}=\frac{\pi a^2}{2}.$$

Another example is the following:



The centre of a sphere, whose radius is a, is on the surface of a right circular cylinder, the radius of whose base is $\frac{a}{2}$. Find the surface of the sphere intercepted by the cylinder.

Take for the equations of the sphere and cylinder,

$$x^2+y^2+z^2=a^2,$$

and $$x^2+y^2=ax.$$

CPAQ is one fourth the required surface. Since this surface is a part of the sphere, the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ must be taken from $x^2+y^2+z^2=a^2$, giving, as in the preceding example,

$$S=\int\int\frac{a\,dx\,dy}{\sqrt{a^2-x^2-y^2}},$$

to be integrated over the region *ORA*, the projection of *CPAQ* on the plane *XY*.

The equation of the curve *ORA* is $x^2+y^2=ax$.

Hence $$\frac{1}{4}S=\int_0^a\int_0^{\sqrt{ax-x^2}}\frac{a\,dx\,dy}{\sqrt{a^2-x^2-y^2}}=\left(\frac{\pi}{2}-1\right)a^2.$$

$$S=(2\pi-4)a^2.$$

Let us now find the surface of the cylinder intercepted by the sphere, one fourth of which is *CPARO*.

Since this is a part of the cylinder $x^2+y^2=ax$, the partial derivatives in (1) must be taken from this equation. But from $x^2+y^2=ax$, we find $\frac{\partial z}{\partial x}=\infty$, $\frac{\partial z}{\partial y}=\infty$.

The formula (1) is, then, inapplicable in this case.

It is also evident from the figure that the surface *CPARO* cannot be found from its projection on the plane *XY*, since this projection is the curve *ORA*.

The difficulty is removed by projecting on the plane *XZ*, and using, instead of (1),

$$S=\int\int\left[1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}}dx\,dz. \quad . \quad . \quad (2)$$

We now find from $x^2+y^2=ax$,

$$\frac{\partial y}{\partial x}=\frac{a-2x}{2y},\quad \frac{\partial y}{\partial z}=0.$$

Substituting in (2), and simplifying,

$$\frac{1}{4}S=\int\int\frac{a\,dx\,dz}{2\sqrt{ax-x^2}}.$$

This must be integrated over the region $CP'AO$, $CP'A$ being the projection on XZ of CPA.

To find the equation of $CP'A$, we eliminate y from

$$x^2+y^2+z^2=a^2 \text{ and } x^2+y^2=ax,$$

giving,

$$z^2=a^2-ax.$$

Hence

$$\frac{1}{4}S=\frac{a}{2}\int_0^a\int_0^{\sqrt{a^2-ax}}\frac{dx\,dz}{\sqrt{ax-x^2}}=a^2,\ S=4\,a^2.$$

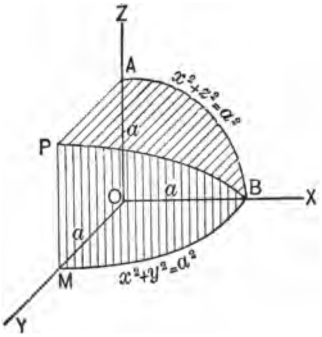
EXAMPLES

1. The axes of two equal right circular cylinders, a being the radius of base, intersect at right angles; find the surface of one intercepted by the other.

Take for the equations of the cylinders,

$$x^2+z^2=a^2, \text{ and } x^2+y^2=a^2.$$

Ans. $8\,a^2$.



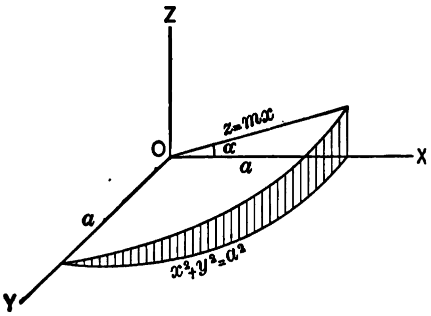
2. Find the area of the part of the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

in the first octant, intercepted by the coördinate planes.

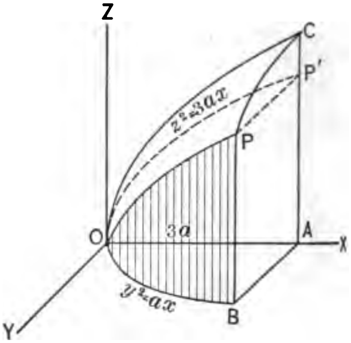
Ans. $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$.

3. Find the area of the surface of the cylinder $x^2+y^2=a^2$, included between the plane $z=mx$ and the plane XY. *Ans.* $4\,ma^2$.



4. Find the area of the surface of the paraboloid of revolution $y^2+z^2=4\,ax$, intercepted by the parabolic cylinder $y^2=ax$, and the plane $x=3\,a$.

Ans. $\dfrac{56\,\pi a^2}{9}$.



5. In the preceding example, find the area of the surface of the cylinder intercepted by the paraboloid of revolution and the given plane.

Ans. $(13\sqrt{13}-1)\dfrac{a^2}{\sqrt{3}}$.

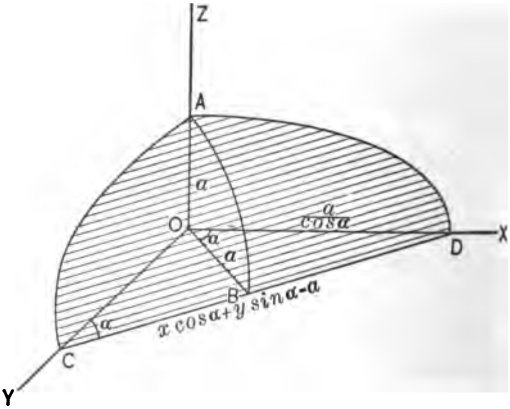
6. Find the area of that part of the surface

$$z^2+(x\cos\alpha+y\sin\alpha)^2=a^2$$

which is situated in the first octant.

The surface is a right circular cylinder, whose axis is the line $z=0$, $x\cos\alpha+y\sin\alpha=0$, and radius of base a.

Ans. $\dfrac{a^2}{\sin\alpha\cos\alpha}$.



7. A diameter of a sphere whose radius is a is the axis of a right prism with a square base, $2\,b$ being the side of the square. Find the surface of the sphere intercepted by the prism.

Ans. $8\,a\left(2\,b\sin^{-1}\dfrac{b}{\sqrt{a^2-b^2}}-a\sin^{-1}\dfrac{b^2}{a^2-b^2}\right)$.

268. To find the Volume of Any Solid bounded by a Surface, whose Equation is given between Three Rectangular Coördinates, x, y, z.

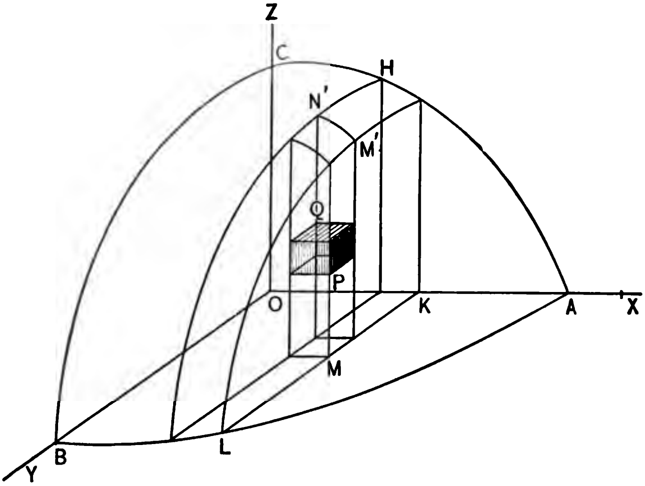
The solid may be supposed to be divided, by planes parallel to the coördinate planes, into elementary rectangular parallelopipeds. The volume of one of these parallelopipeds is $dx\,dy\,dz$, and the volume of the entire solid is

$$V=\int\int\int dx\,dy\,dz,$$

the limits of the integration depending upon the equation of the bounding surface.

For example, let us find the volume of one eighth of the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$



PQ represents one of the elementary parallelopipeds whose volume is $dx\,dy\,dz$.

If we integrate with respect to z, we collect all the elements in the column MN', z varying from zero to MM'; that is,

$$\text{from } 0 \text{ to } z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

Integrating next with respect to y, we collect all the columns in the slice $KLN'H$, y varying from zero to KL; that is,

$$\text{from } 0 \text{ to } y = b\sqrt{1-\frac{x^2}{a^2}}.$$

This value of y is taken from the equation of the curve ALB, which is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Finally, we integrate with respect to x, to collect all the slices in the entire solid ABC. Here x varies from zero to OA; that is, from 0 to a.

The y and x integrations are said to be over the region AOB.

Hence we have $V=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dx\,dy\,dz=\frac{\pi abc}{6}.$

For the entire ellipsoid $V=\frac{4\pi abc}{3}.$

EXAMPLES

1. Find the volume of one of the wedges cut from the cylinder $x^2+y^2=a^2$ by the plane $z=x\tan\alpha$ and the plane XY. (See Figure, Ex. 3, Art. 267.)

Ans. $2\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{x\tan\alpha}dx\,dy\,dz=\frac{2a^3\tan\alpha}{3}.$

2. Find the volume of the tetrahedron bounded by the coördinate planes and by the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Ans. $\frac{abc}{6}.$

3. Find the volume included between the paraboloid of revolution $y^2+z^2=4ax$, the parabolic cylinder $y^2=ax$, and the plane $x=3a$. (See Figure, Ex. 4, Art. 267.) *Ans.* $(6\pi+9\sqrt{3})a^3.$

4. Find the volume contained between the paraboloid of revolution $x^2+y^2=az$, the cylinder $x^2+y^2=2ax$, and the plane XY.

Ans. $\dfrac{3\pi a^3}{2}$.

5. Find the volume of the cylinder $x^2+y^2=ax$, intercepted by the paraboloid of revolution $y^2+z^2=2ax$.

$\left(\dfrac{\pi}{4}+\dfrac{2}{3}\right)a^3$.

6. The centre of a sphere (radius a) is on the surface of a right circular cylinder, the radius of whose base is $\dfrac{a}{2}$. Find the volume of the part of the cylinder intercepted by the sphere. (See second Figure, Art. 267.)

Ans. $\dfrac{2}{3}\left(\pi-\dfrac{4}{3}\right)a^3$.

7. Find the volume in the first octant, bounded by the surface

$$\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1.$$

Ans. $\dfrac{abc}{90}$.

8. Find the entire volume within the surface

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

Ans. $\dfrac{4\pi a^3}{35}$.

269. Moment of Inertia of Any Solid. This may be expressed by a triple integral.

Thus, the moment of inertia about OX, m being the mass of a unit of volume, is

$$I=m\int\int\int(y^2+z^2)\,dx\,dy\,dz,$$

with similar formulæ for the moments of inertia about the axes OY, OZ.

EXAMPLES

1. Find the moment of inertia about OX of the rectangular parallelopiped bounded by the planes $x=a$, $y=b$, $z=c$, and the coördinate planes.

Ans. $(b^2+c^2)\dfrac{mabc}{3}$.

2. Find the moment of inertia about OZ of the tetrahedron bounded by the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

and by the coördinate planes. *Ans.* $(a^2+b^2)\frac{mabc}{60}$.

3. Find the moment of inertia about OX of the portion of the cylinder $x^2+y^2=a^2$ included between the planes $z=h$ and $z=-h$.

Ans. $\pi ma^2h\left(\frac{a^2}{2}+\frac{2h^2}{3}\right)$.

4. Find the moment of inertia of the preceding cylinder about OZ.

Ans. πma^4h.

5. Find the moment of inertia of a sphere (radius a) about a diameter. *Ans.* $\frac{8\pi ma^5}{15}$.

6. Find the moment of inertia about OZ of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad \textit{Ans.} \quad \frac{4\pi mabc}{15}(a^2+b^2)$$

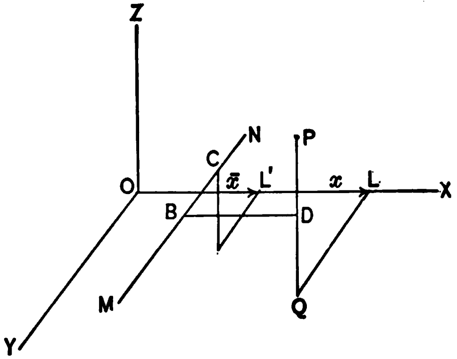
CHAPTER XXXII

CENTRE OF GRAVITY. PRESSURE OF FLUIDS. FORCE OF ATTRACTION

CENTRE OF GRAVITY

270. Definition. The centre of gravity of a body is a point so situated that the force of gravity acting on the body produces no tendency to rotate about an axis passing through the point.

271. Coördinates of Centre of Gravity. To find the centre of gravity, C, of any body, take P as any infinitesimal part of the given body, PQ the line of direction of gravity, and MN any horizontal axis passing through C. Let BD be the common perpendicular between MN and PQ. Take the axis of X parallel to BD and represent by x and $\bar{x}$, OL and OL', the x coördinates of P and C respectively. Then the distance $BD = L'L = x - \bar{x}$.



The force exerted by gravity on P is proportional to and therefore may be measured by its mass. Denoting its mass by dm, the moment of this force about MN would be $(x - \bar{x})\,dm$; and if dm is an infinitesimal in one, two, or three dimensions, the tendency of the whole body to rotate about MN is equal to $\int (x - \bar{x})dm$.

Since this must equal zero,

$$\int (x-\bar{x})dm = 0,$$

and

$$\bar{x} = \frac{\int x dm}{\int dm}. \quad . \quad . \quad . \quad . \qquad (1)$$

Similar formulæ may be derived for $\bar{y}$ and $\bar{z}$.

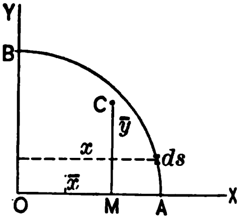
Note. — The mass of a unit's volume is called *density*. If we represent the density by ρ, the differential mass or dm is equal to ρ multiplied by the differential of the arc, area, or volume.

Ex. 1. Find the centre of gravity of a quarter of the arc of a circle

Let the equation of the circle be $x^2+y^2=a^2$.

Here $dm = \rho ds$.

Substituting in (1), Art. 271, we have



$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{a\int_0^a x(a^2-x^2)^{-\frac{1}{2}}\,dx}{\frac{1}{2}\pi a} = \frac{2a}{\pi}.$$

From the symmetry of the figure $\bar{y} = \frac{2a}{\pi}$.

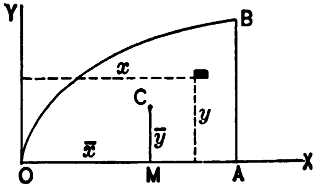
Ex. 2. Find the centre of gravity of the surface bounded by a parabola, its axis, and one of its ordinates.

Let the equation of the parabola be $y^2 = 4px$, B being $(9p, 6p)$.

Here $dm = \rho\,dx\,dy$, and substituting in formula (1), Art. 271.

$$\bar{x} = \frac{\int_0^{9p}\int_0^{\sqrt{4px}} x\,dx\,dy}{\int_0^{9p}\int_0^{\sqrt{4px}} dx\,dy}$$

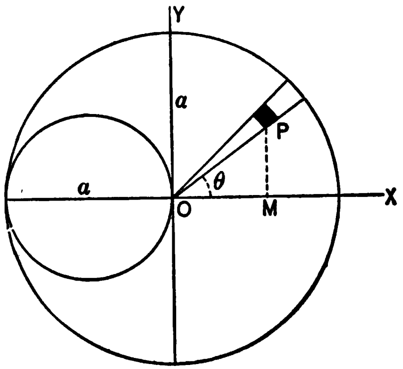
$$= \frac{\sqrt{4p}\int_0^{9p} x^{\frac{3}{2}}\,dx}{\sqrt{4p}\int_0^{9p} x^{\frac{1}{2}}\,dx} = \frac{27}{5}p.$$



Similarly,

$$\bar{y}=\frac{\int_0^{9p}\int_0^{\sqrt{4px}} y\,dx\,dy}{\int_0^{9p}\int_0^{\sqrt{4px}} dx\,dy}=\frac{2p\int_0^{9p} x\,dx}{\sqrt{4p}\int_0^{9p} x^{\frac{1}{2}}\,dx}=\frac{9p}{4}.$$

Ex. 3. Find the centre of gravity of a circular disk of radius a, whose density varies directly as the distance from the centre, and from which a circle described upon a radius as a diameter has been cut.



Let the equation of the large circle be $r=a$; and the equation of the small circle be $r=-a\cos\theta$.

The disk is symmetrical with respect to OX, hence $\bar{y}=0$.

Here

$dm=\rho r\,d\theta\,dr=\kappa r^2\,d\theta\,dr$,

(if $\rho=\kappa r$).

Also $x=OM=r\cos\theta$.

$$\text{Therefore } \bar{x}=\frac{\int_0^{\frac{\pi}{2}}\int_0^{a} r^3\cos\theta\,d\theta\,dr+\int_{\frac{\pi}{2}}^{\pi}\int_{-a\cos\theta}^{a} r^3\cos\theta\,d\theta\,dr}{\int_0^{\frac{\pi}{2}}\int_0^{a} r^2\,d\theta\,dr+\int_{\frac{\pi}{2}}^{\pi}\int_{-a\cos\theta}^{a} r^2\,d\theta\,dr}.$$

$$\bar{x}=\frac{\frac{a^4}{4}\left[\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta+\int_{\frac{\pi}{2}}^{\pi}(\cos\theta-\cos^5\theta)\,d\theta\right]}{\frac{a^3}{3}\left[\int_0^{\frac{\pi}{2}}d\theta+\int_{\frac{\pi}{2}}^{\pi}(1+\cos^3\theta)\,d\theta\right]}$$

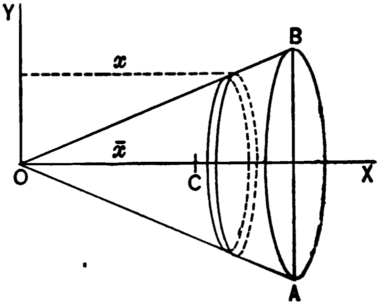
$$=\frac{6a}{5(3\pi-2)}=0.1616\,a.$$

Ex. 4. Find the centre of gravity of a cone of revolution, the radius of the base being 2 and the altitude 6.

The equation of OB is $y = \frac{1}{3}x$.

Here $dm = \rho\pi y^2\,dx$, and substituting in (1), Art. 271,

$$\bar{x} = \frac{\int_0^6 xy^2\,dx}{\int_0^6 y^2\,dx} = \frac{9}{2}.$$



The cone is symmetrical with respect to OX, hence $\bar{y} = 0$.

Note.—On comparing the formulæ for the centre of gravity of arc, area, and volume,

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{x} = \frac{\int x\,dA}{\int dA}, \quad \bar{x} = \frac{\int x\,dV}{\int dV},$$

we notice that, in each case, the element of the numerator integral is x times the element of the denominator integral.

5. Find the centre of gravity of the arc of the hypocycloid (Art. 132) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in the first quadrant. *Ans.* $\bar{x} = \bar{y} = \frac{2}{5}a$.

6. Find the centre of gravity of the arc of one arch of the cycloid (Art. 284)

$$x = a \operatorname{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}. \qquad \textit{Ans. } \bar{x} = \pi a,\ \bar{y} = \frac{4}{3}a.$$

7. Find the centre of gravity of a straight rod of length a, the density of which varies as the third power of the distance of each point from one end. *Ans.* $\bar{x} = \frac{4}{5}a$.

8. Find the centre of gravity of the surface of a hemisphere when the density at each point of the surface varies as its perpendicular distance from the base of the hemisphere. *Ans.* $\bar{x} = \frac{2a}{3}$.

9. Find the centre of gravity of a semiellipse. *Ans.* $\bar{x}=\frac{4a}{3\pi}$.

10. Find the centre of gravity of the area between the cissoid $y^2=\frac{x^3}{2a-x}$ and its asymptote. *Ans.* $\bar{x}=\frac{5}{3}a$.

11. Find the centre of gravity of the area bounded by the parabola $y^2=8x$, the line $y+x-6=0$, and the axis of X.

Ans. $\bar{x}=\frac{62}{25}$; $\bar{y}=\frac{7}{5}$.

12. Find the centre of gravity of one loop of the curve $r=a\sin 2\theta$.

Ans. $\bar{x}=\frac{128a}{105\pi}$; $\bar{y}=\frac{128a}{105\pi}$.

13. Find the centre of gravity of the upper half of the cardioid $r=a(1-\cos\theta)$.

Ans. $\bar{x}=-\frac{5}{6}a$;

$\bar{y}=\frac{16a}{9\pi}=.57a$.

14. Find the centre of gravity of one loop of the lemniscate $r^2=a^2\cos 2\theta$. *Ans.* $\bar{x}=\frac{\pi\sqrt{2}}{8}a=.55a$.

15. Find the centre of gravity of a hemisphere. *Ans.* $\bar{x}=\frac{3}{8}a$.

16. Find the centre of gravity of a hemispheroid. *Ans.* $\bar{x}=\frac{3}{8}a$.

17. A right cone of height h is scooped out of a right cylinder of the same height and base. Find the distance of the centre of gravity of the remainder from the vertex. *Ans.* $\frac{3}{8}h$.

272. Theorems of Pappus.

Theorem I. If a plane area be revolved about an axis in its plane and not crossing the area, the volume of the solid generated is equal to the product of the area and the length of the path described by the centre of gravity of the area.

Theorem II. If the arc of a curve be revolved about an axis in its plane and not crossing the arc, the area of the surface generated is equal to the product of the length of the arc and the path described by the centre of gravity of the arc.

273. Proof of the Theorems. Let the area be in the plane XY and let it revolve about the axis of X. Then by (1), Art. 271, we have

$$\bar{y} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy}.$$

Or $$\bar{y}\int\int dx\,dy = \int\int y\,dx\,dy.$$

Then $$2\,\pi\bar{y}\int\int dx\,dy = \int\int 2\,\pi y\,dx\,dy. \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

But the right-hand member of equation (1) is the volume described by revolving the area through the angle $2\,\pi$, $2\,\pi\bar{y}$ is the length of the path described by the centre of gravity, and $\int\int dx\,dy$ is the plane area.

The first theorem is thus seen to be true, and the second can be proved true in a similar manner.

EXAMPLES

1. Find the volume and surface generated by revolving a rectangle with dimensions a and b about an axis c units from the centre of the rectangle.

Ans. $2\,\pi abc$, and $4\,\pi(a+b)c$.

2. Find the volume and surface generated by revolving an equilateral triangle each side a units in length, about an axis c units from the centre of the triangle.

Ans. $\frac{\pi a^2c\sqrt{3}}{2}$ and $6\,\pi ac$.

3. Find the volume and surface generated by revolving a circle of radius a about an axis b units from the centre of the circle.

Ans. $2\pi^2a^2b$ and $4\pi^2ab$.

4. Find the volume generated by revolving an ellipse, semiaxes a and b, about an axis c units from the centre of the ellipse.

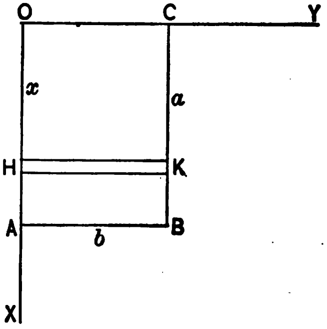
Ans. $2\pi^2abc$.

PRESSURE OF LIQUIDS

274. The pressure of a liquid on any given horizontal surface is equal to the weight of a column of the liquid whose base is the given surface and whose height is equal to the distance of this surface below the surface of the liquid.

The pressure on any vertical surface varies as the depth, and the method of determining it is illustrated by the following examples.

Ex. **1.** Suppose it is required to find the pressure on the rectangular board $OABC$, the edge OC being at the surface of the water.



Let $BC = a$, and $AB = b$.

Suppose the rectangle divided into horizontal strips one of which is HK.

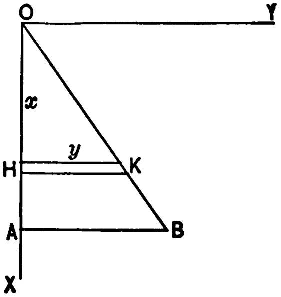
Let $OH = x$, then the width of the strip is dx.

If the pressure on this strip were uniform throughout and the same as it is at the top of the strip, the pressure on the strip would be $wbx \cdot dx$, where w is the weight of a cubic unit of the water. And the entire pressure on the board is evidently the integral of this expression.

That is, Entire pressure $= \int_0^a wbx\,dx = \frac{a^2bw}{2}$.

Ex. 2. Find the pressure on that part of the board in Example 1, which is below the diagonal.

In this case the area of HK is $y\,dx$, and the entire pressure on the triangular board is



$$\int_0^a wyx\,dx.$$

But $$y=\frac{b}{a}x,$$

hence entire pressure

$$=\frac{bw}{a}\int_0^a x^2dx=\frac{bwa^2}{3}.$$

Note. — Call the weight of a cubic foot of water, 62½ lbs.

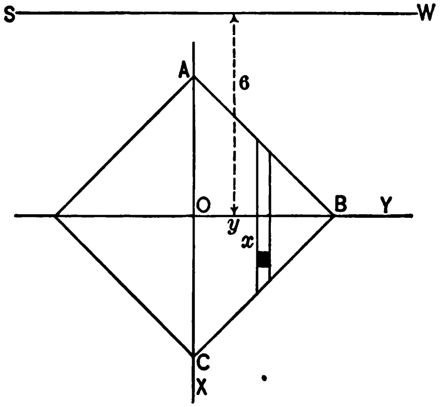
Ex. 3. One face of a box immersed in water is in the form of a square, the diagonals being 8 feet in length. The centre of the square is 6 feet below the surface of the water, and one diagonal is vertical. Find the pressure on the square face.

Let SW be the surface of the water. Taking the axes as in the figure, the equations of AB and BC are $y=4+x$, and $y=4-x$, respectively.

Then, if P represents the entire pressure on the board,

$$P=2w\int_0^4\int_{y-4}^{4-y}(6+x)dy\,dx$$

$$=192\,w=12000 \text{ lbs.}$$

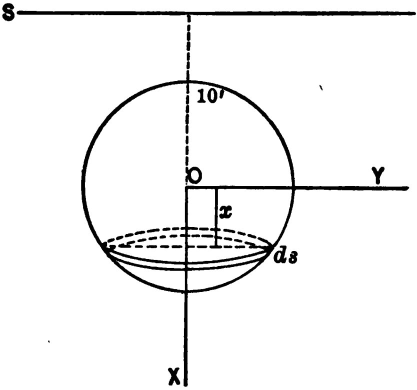


Ex. 4. Find the pressure on a sphere 6 feet in diameter, immersed in water, the centre of the sphere being 10 feet below the surface of the water.

Let SW be the surface of the water.

Take the axes as in the figure, and let the elementary surface be a zone. The area of a zone at a distance x from the centre of the sphere is $2\pi y\,ds$. The pressure on the zone is $2\pi wy(10+x)ds$.

Then, if P represents the entire pressure on the sphere,



$$P = 2\pi w\int y(10+x)ds.$$

But $$y=\sqrt{9-x^2}, \text{ and } ds = \frac{3}{y}\,dx.$$

Hence $$P = 6\pi w\int_{-3}^{3}(10+x)dx,$$

$$= 360\,\pi w = 22500\pi \text{ lbs.}$$

5. A rectangular flood gate whose upper edge is in the surface of the water, is divided into three parts by two lines from the middle of lower edge to the extremities of upper edge. Show that the parts sustain equal pressures.

6. A rectangular flood gate 10 feet broad and 6 feet deep has its upper edge in the surface of the water. How far must it be sunk to double the pressure? *Ans.* 3 ft.

7. A board in the form of a parabolic segment by a chord perpendicular to the axis is immersed in water. The vertex is at the surface and the axis vertical. It is 20 feet deep and 12 feet broad. Find the pressure in tons. *Ans.* 60

8. How far must the board in Ex. 7 be sunk to double the pressure? *Ans.* 12 ft.

9. Suppose the position of the parabolic board in Ex. 7 reversed, the chord being in the surface; what is the pressure? *Ans.* 40 tons.

10. How far must the board in Ex. 9 be sunk to double the pressure? *Ans.* 8 ft.

11. A trough 2 feet deep and 2 feet broad at the top has semi-elliptical ends. If it is full of water, find the pressure on one end. *Ans.* $166\frac{2}{3}$ lbs.

12. One end of an unfinished water main 2 feet in diameter is closed by a temporary bulkhead and the water is let in from the reservoir. Find the pressure on the bulkhead if its centre is 30 feet below the surface of the water in the reservoir. *Ans.* 1875π lbs.

13. A water tank is in the form of a hemisphere 24 feet in diameter surmounted by a cylinder of the same diameter and 10 feet high. Find the total pressure on the surface of the tank when the tank is filled to within 2 feet of the top. *Ans.* $150\,\pi$ tons.

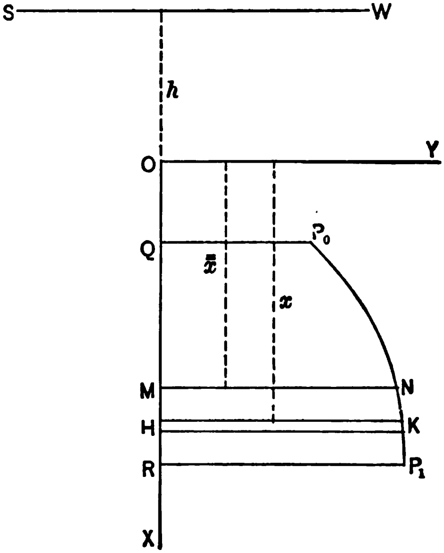
14. A cylindrical vessel, whose depth is 12 inches and base a circle of 20 inches diameter, is filled with equal parts of water and oil. Assuming the oil to be half as heavy as the water, show that the pressure on the base equals the lateral pressure.

275. Centre of Pressure. Since the pressure of a liquid on a vertical surface varies as the depth, there exists a horizontal line about which the statical moment of the entire pressure on the surface is zero. Such a line passes through the *centre of pressure* and the abscissa of this point may be found by the method used in the following example.

Ex. 1. Find the abscissa of the centre of liquid pressure on a vertical surface bounded by the curve $y = f(x)$, the axis of X and the two ordinates y_0 and y_1. Given that the origin is at a distance

h below the surface of the liquid, the axis of X vertical, and the weight of a cubic unit of liquid is w.

Let P_0P_1RQ be the surface bounded by the curve $y = f(x)$, the axis of X, and the two ordinates $y_0 = QP_0$ and $y_1 = RP_1$. Divide the surface into horizontal strips of width dx, one of which is HK. Let $OH = x$. Let MN pass through the centre of liquid pressure, and $OM = \bar{x}$.



Then the pressure on the strip HK is $wy(h + x)\,dx$, and the moment of this pressure about MN is

$$wy(h + x)(x - \bar{x})\,dx.$$

Therefore, the moment of the entire pressure is the integral of this expression between the abscissas of P_0 and P_1, that is, between x_0 and x_1. But this must equal zero, therefore

$$\int_{x_0}^{x_1} wy(h+x)(x-\bar{x})\,dx = 0.$$

Or $$\bar{x} = \frac{\int_{x_0}^{x_1} xy(h+x)\,dx}{\int_{x_0}^{x_1} y(h+x)\,dx}.$$

2. Find the centre of pressure of the water on the parabolic board given in Ex. 7, Art. 274. *Ans.* $14\frac{2}{7}$ ft. below vertex.

3. Find the centre of pressure of the water on the bulkhead given in Ex. 12, Art. 274. *Ans.* $\frac{1}{10}$ inch below centre of bulkhead.

4. A rectangular flood gate a feet deep and b feet broad, with its upper edge at the surface, is to be braced along a horizontal line. How far down must the brace be put that the gate may not tend to turn about it? *Ans.* $\frac{2}{3}\,a$ ft.

5. One end of a cylindrical aqueduct 6 feet in diameter which is half full of water is closed by a water-tight bulkhead held in place by a brace. How far below the centre of the bulkhead should the brace be put? What pressure must it be able to withstand?

Ans. $\bar{\bar{x}} = \frac{9}{16}\pi$ ft.; $P = 1125$ lbs.

6. A water pipe passes through a masonry dam, enters a reservoir, and is closed by a cast-iron circular valve which is hinged at the top. The diameter of the valve is 3 feet, and the depth of its centre below the water level in the reservoir is 12 feet. Find the pressure on the valve, and the distance of the centre of pressure below the hinge. *Ans.* $P = 1687\frac{1}{2}\pi$ lbs. and $\frac{99}{64}$ ft.

7. Water is flowing along a ditch of rectangular section 4 feet deep and 1 foot wide. The water is stopped by a board fitting the ditch and held vertical by two bars crossing the ditch horizontally, one at the bottom and the other one foot from the bottom of the ditch. How high must the water rise to force a passage by upsetting the board? *Ans.* To within 1 ft. of top of ditch.

ATTRACTION AT A POINT

276. A particle of mass m is situated at a perpendicular distance c from one end of a thin, straight, homogeneous wire of mass M and length l. Required to find the attraction on the particle due to the wire.

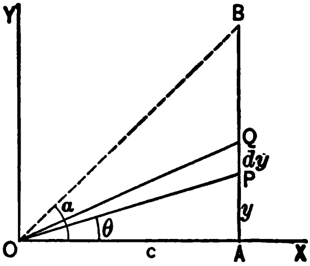
Let O be the particle and AB the wire. Let X and Y be the components of the attraction along the axes of X and Y respectively.

Divide AB into elements of length dy and let PQ be one of these elements.

The mass of PQ is $\frac{M}{l}dy$, since $\frac{M}{l}$ is the mass of a unit's length.

If the mass of PQ were concentrated at P, the attraction at O due to PQ is, according to Newton's Law of Attraction,

$$\frac{\kappa m M\,dy}{l(c^2+y^2)},$$



and the components along OX and OY are

$$\frac{\kappa m M\,dy}{l(c^2+y^2)}\cos\theta,\ \text{and}\ \frac{\kappa m M\,dy}{l(c^2+y^2)}\sin\theta,$$

respectively.

Substituting for $\cos\theta$ and $\sin\theta$ their values we have

$$X=\frac{\kappa m M c}{l}\int_0^l\frac{dy}{(c^2+y^2)^{\frac{3}{2}}}=\frac{\kappa m M}{c\sqrt{c^2+l^2}}=\frac{\kappa m M}{cl}\sin\alpha.$$

$$Y=\frac{\kappa m M}{l}\int_0^l\frac{y\,dy}{(c^2+y^2)^{\frac{3}{2}}}=\frac{\kappa m M}{cl}\left[1-\frac{c}{\sqrt{c^2+l^2}}\right]$$

$$=\frac{\kappa m M}{cl}(1-\cos\alpha),$$

where the angle $AOB=\alpha$.

Denoting by R the total attraction of the wire on the particle,

$$R=\sqrt{X^2+Y^2}=\frac{\kappa m M}{cl}\sqrt{2(1-\cos\alpha)}$$

$$=\frac{2\,\kappa m M}{cl}\sin\frac{1}{2}\alpha.$$

The line of attraction evidently makes with OA an angle whose tangent is

$$\frac{Y}{X}=\frac{1-\cos\alpha}{\sin\alpha}=\tan\frac{1}{2}\alpha.$$

The resultant attraction, therefore, bisects the angle α.

NOTE. — If we take as our *unit of force* the force of attraction between two unit masses concentrated at points which are at unit's distance apart, κ becomes unity.

EXAMPLES

1. Find the attraction perpendicular to the wire in preceding example when the particle is at a distance $\frac{l}{3}$ above O.

Ans. $\frac{\kappa m M}{c}\left[\frac{2}{\sqrt{9c^2+4l^2}}+\frac{1}{\sqrt{9c^2+l^2}}\right]$.

2. Find the attraction of a thin, straight, homogeneous wire of length l and mass M upon a particle or mass m which is situated at a distance c from one end of the wire and in its line of direction.

Ans. $\frac{\kappa m M}{c(c+l)}$.

3. Find the attraction of a homogeneous circular disk of radius a upon a particle of mass m in its axis and at a distance c from the disk.

Ans. $2\kappa\pi m\rho\left[1-\frac{c}{\sqrt{c^2+a^2}}\right]$, where ρ is the mass of the disk corresponding to unit's surface.

4. Find the attraction due to a homogeneous right circular cylinder of length $2l$ and radius a upon a mass m in the axis produced of the cylinder and distant c from one end.

Ans. $2\pi\kappa m\rho\,[2l+\sqrt{a^2+c^2}-\sqrt{a^2+(c+2l)^2}]$, where ρ is the mass of unit's volume of the cylinder.

CHAPTER XXXIII

HYPERBOLIC FUNCTIONS. CYCLOID, TROCHOID, EPICYCLOID, AND HYPOCYCLOID

This chapter includes certain subjects involving both the Differential and Integral Calculus.

HYPERBOLIC FUNCTIONS

277. Definition. There are certain combinations of exponential functions called *hyperbolic functions*, having relations to each other analogous to those between the trigonometric functions.

They are defined as follows:

$\dfrac{e^x - e^{-x}}{2}$ is called the hyperbolic sine of x, and written $\sinh x$.

$\dfrac{e^x + e^{-x}}{2}$ is called the hyperbolic cosine of x, and written $\cosh x$.

Thus

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Also for the hyperbolic tangent, cotangent, secant, and cosecant of x,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

It is evident from the definitions that

$$\sinh 0 = 0, \qquad \cosh 0 = 1,$$
$$\sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x.$$

278. Relations between Hyperbolic Functions. From the definitions in the preceding article, we find

$$\cosh^2 x - \sinh^2 x = 1,$$
$$\tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{cosech}^2 x = 1,$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y},$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

279. Inverse Hyperbolic Functions.

If $\qquad x = \sinh y, \quad \ldots \ldots \ldots \quad (1)$

then $\qquad y = \sinh^{-1} x.$

This function may be expressed as a logarithm as follows:

From (1) $\qquad x = \frac{e^y - e^{-y}}{2}.$

Solving this with reference to y,

$$y = \log(x + \sqrt{x^2 + 1}).$$

Hence $\qquad \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}).$

Similarly, $\qquad \cosh^{-1} x = \pm \log(x + \sqrt{x^2 - 1}).$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x},$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x} = \frac{1}{2}\log\frac{x+1}{x-1},$$

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x} = \pm\log\frac{1+\sqrt{1-x^2}}{x}.$$

$$\operatorname{cosech}^{-1} x = \sinh^{-1}\frac{1}{x} = \log\frac{1+\sqrt{1+x^2}}{x}.$$

280. Differentiation of Hyperbolic Functions. From the definitions we have

$$\frac{d}{dx}\sinh x = \cosh x,$$

$$\frac{d}{dx}\cosh x = \sinh x,$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x,$$

$$\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x,$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x,$$

$$\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x.$$

To differentiate the inverse function

$$y = \sinh^{-1} x,$$

we have $x = \sinh y,$

$$\frac{dx}{dy} = \cosh y = \sqrt{\sinh^2 y + 1} = \sqrt{x^2+1}.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}.$$

Hence $$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}.$$

Similarly,
$$\frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{x^2-1}},$$
$$\frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2},$$
$$\frac{d}{dx}\coth^{-1}x=\frac{1}{1-x^2},$$
$$\frac{d}{dx}\operatorname{sech}^{-1}x=-\frac{1}{x\sqrt{1-x^2}},$$
$$\frac{d}{dx}\operatorname{cosech}^{-1}x=-\frac{1}{x\sqrt{1+x^2}}.$$

281. Inverse Circular, and Inverse Hyperbolic, Functions as Integrals. A comparison of the integrals involving the inverse circular functions, with those involving the inverse hyperbolic functions, shows the close analogy between them.

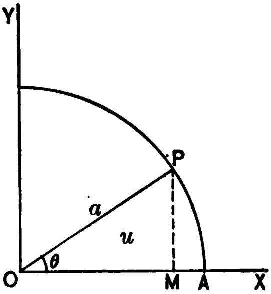
$$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}, \qquad \int\frac{dx}{\sqrt{x^2+a^2}}=\sinh^{-1}\frac{x}{a}.$$
$$\text{or } =-\cos^{-1}\frac{x}{a}. \qquad \int\frac{dx}{\sqrt{x^2-a^2}}=\cosh^{-1}\frac{x}{a}.$$
$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}, \qquad \int\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a},$$
$$\text{or } =-\frac{1}{a}\cot^{-1}\frac{x}{a}. \qquad \text{or } =\frac{1}{a}\coth^{-1}\frac{x}{a}.$$
$$\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}, \qquad \int\frac{dx}{x\sqrt{a^2-x^2}}=-\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a}.$$
$$\text{or } =-\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}. \qquad \int\frac{dx}{x\sqrt{a^2+x^2}}=-\frac{1}{a}\operatorname{cosech}^{-1}\frac{x}{a}.$$

282. Circular and Hyperbolic Functions as related to the Circle and Equilateral Hyperbola. The hyperbolic functions are so called because their relation to the equilateral hyperbola corresponds to that of the circular functions to the circle.

Let us express x and y, the coördinates of a point P of the circle

$$x^2+y^2=a^2,$$

in terms of the area of the circular sector *POA*.



Calling the area of the sector u, and θ the angle *POA*, we have

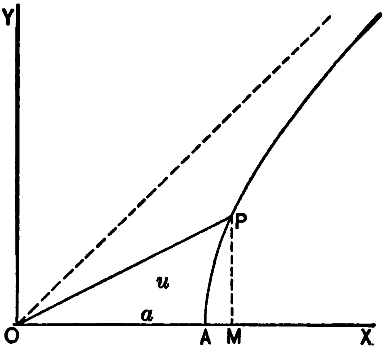
$$u=\frac{a^2\theta}{2},\quad x=a\cos\theta,\quad y=a\sin\theta.$$

Hence $x=a\cos\dfrac{2u}{a^2},\quad y=a\sin\dfrac{2u}{a^2}.\quad(1)$

Now if x and y are the coördinates of a point P of the equilateral hyperbola

$$x^2-y^2=a^2, \quad . \quad . \quad . \quad (2)$$

and u the area of the hyperbolic sector *POA*, it will be shown that equations (1) will still hold, provided that "sin" and "cos" are changed into "sinh" and "cosh."



Here the area *POA* is

$$u=\frac{a^2}{2}\log\frac{x+y}{a},$$

(See Ex. 5, p. 323.)

whence $$\frac{x+y}{a}=e^{\frac{2u}{a^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3), $$\frac{x-y}{a}=e^{-\frac{2u}{a^2}}.$$

Hence $$\frac{x}{a}=\frac{e^{\frac{2u}{a^2}}+e^{-\frac{2u}{a^2}}}{2}=\cosh\frac{2u}{a^2},\quad \frac{y}{a}=\frac{e^{\frac{2u}{a^2}}-e^{-\frac{2u}{a^2}}}{2}=\sinh\frac{2u}{a^2};$$

whence $x=a\cosh\dfrac{2u}{a^2},\quad y=a\sinh\dfrac{2u}{a^2}$, as was to be proved.

283. Exercises in Hyperbolic Functions.

1. $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$.

2. $\cosh 3x = -3 \cosh x + 4 \cosh^3 x$.

3. $\tanh^{-1} \sin x = \operatorname{sech}^{-1} \cos x$.

4. $\tan^{-1} \sinh x = \sec^{-1} \cosh x$.

5. $\tan^{-1} \sinh 2x = 2 \tan^{-1} \tanh x$.

6. $\tanh^{-1} \sin 2x = 2 \tanh^{-1} \tan x$.

7. $\cos^{-1} \cosh 2x = 2 \cos^{-1} \cosh x$.

8. $\cosh^{-1} \cos 2x = 2 \cosh^{-1} \cos x$.

9. $y = \tan^{-1} x + \tanh^{-1} x$, $\qquad \frac{dy}{dx} = \frac{2}{1 - x^4}$.

10. $y = \log \cosh x$, $\qquad \frac{dy}{dx} = \tanh x$.

11. $y = \sinh x \sin x + \cosh x \cos x$, $\qquad \frac{dy}{dx} = 2 \sinh x \cos x$.

12. $y = \tan^{-1} \tanh x$, $\qquad \frac{dy}{dx} = \operatorname{sech} 2x$.

13. $y = \sinh^{-1} \tan x$, $\qquad \frac{dy}{dx} = \sec x$.

14. $y = \sin^{-1} \tanh x + \tan^{-1} \sinh x$, $\qquad \frac{dy}{dx} = 2 \operatorname{sech} x$.

15. $y = \tan^{-1} \sqrt{\tanh x} + \tanh^{-1} \sqrt{\tanh x}$, $\qquad \frac{dy}{dx} = \sqrt{\coth x}$.

16. Express the equation of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, (Art. 128) and also the length of the arc from the vertex, in hyperbolic functions.

Ans. $y = a \cosh \frac{x}{a}$, and $s = a \sinh \frac{x}{a}$.

17. Show that the equation of the catenary, referred to its vertex as origin, is $\qquad y = 2a \sinh^2 \frac{x}{2a}$.

EQUATIONS AND PROPERTIES OF THE CYCLOID

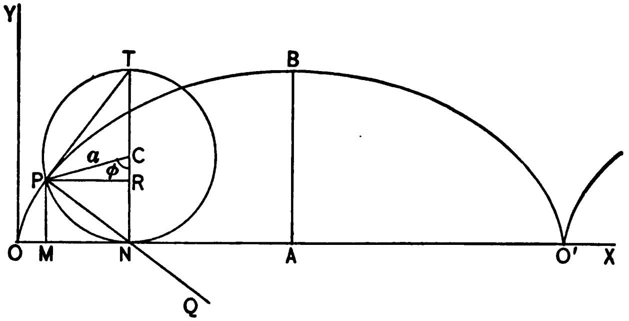
284. Definition and Equation. The cycloid is the curve described by a point in the circumference of a circle, as it rolls along a straight line.

Let OX be the straight line. As the circle NPT, with radius a, rolls along this line, the point P describes the cycloid OBO'.

Let the angle through which the circle has rolled from O,

$$PCN = \phi;$$

and let x, y, be the coördinates of P.



As the arc PN has rolled over the line ON, it is evident that

$$ON = \text{arc } PN = a\phi.$$

Then
$$\left.\begin{aligned} x &= OM = ON - PR = a\phi - a\sin\phi, \\ y &= PM = CN - CR = a - a\cos\phi, \end{aligned}\right\} \quad . \; . \; . \; . \quad (1)$$

If we eliminate ϕ between these equations, we have

$$x = a\cos^{-1}\frac{a-y}{a} - \sqrt{2ay - y^2},$$

or
$$x = a\,\text{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}. \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

This is the equation of the cycloid, but the parametric equations (1) are generally more useful than (2).

285. Cycloid referred to Vertex. The point B is called the vertex of the curve. If the origin is transferred from O to B with parallel axes, we have, x', y', being the new coördinates,

$$y = y' + 2a, \qquad x = x' + \pi a.$$

Substituting these in (1) of the preceding article, we obtain

$$\begin{aligned} x' &= a(\phi - \pi) - a \sin \phi, \\ y' &= -a - a \cos \phi. \end{aligned}$$

Letting $\phi - \pi = \phi'$, the angle through which the circle has rolled from A, and omitting the accents on x' and y', we have

$$\left. \begin{aligned} x &= a\phi' + a \sin \phi', \\ y &= -a + a \cos \phi', \end{aligned} \right\} \quad \text{. . . . . .} \quad (1)$$

the equation of the cycloid referred to its vertex.

286. Tangent and Normal. From (1) Art. 284, we have

$$\frac{dx}{d\phi} = a(1 - \cos \phi) = 2a \sin^2 \frac{\phi}{2}, \quad \text{. . . . . .} \quad (1)$$

$$\frac{dy}{d\phi} = a \sin \phi = 2a \sin \frac{\phi}{2} \cos \frac{\phi}{2};$$

therefore $$\frac{dy}{dx} = \cot \frac{\phi}{2}. \quad \text{. . . . .} \quad (2)$$

But $$\text{angle } PTN = \tfrac{1}{2} PCN = \frac{\phi}{2};$$

hence $$\frac{dy}{dx} = \cot PTN = \tan TPR.$$

Consequently PT is the tangent to the curve at P, and PN the normal.

287. Radius of Curvature. From (1) and (2) of the preceding article, we find

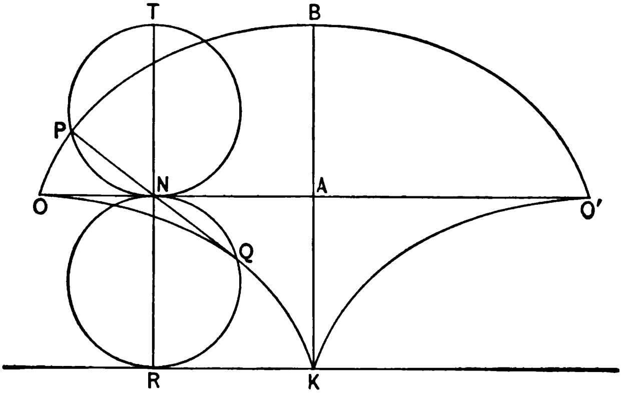
$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 \frac{\phi}{2} \cdot \frac{1}{2} \frac{d\phi}{dx} = -\frac{\operatorname{cosec}^2 \frac{\phi}{2}}{4a \sin^2 \frac{\phi}{2}} = -\frac{1}{4a \sin^4 \frac{\phi}{2}}.$$

Substituting in the expression for the radius of curvature, we have

$$\rho = -\left(1 + \cot^2 \frac{\phi}{2}\right)^{\frac{3}{2}} 4\,a \sin^4 \frac{\phi}{2} = -4a \sin \frac{\phi}{2} = -2\,PN.$$

Hence, if we produce PN to Q, making $NQ = PN$, Q will be the centre of curvature for the point P.

288. Evolute. Produce the diameter TN, making $NR = TN$, and on NR as diameter describe a circle. This circle will pass through Q, since $NQ = PN$.



The $\qquad$ arc NQ = arc $PN = ON$,

and $\qquad$ arc $NQR = OA$;

therefore $\qquad$ arc $QR = OA - ON = RK$.

Hence Q is a point in an equal cycloid, generated by rolling the circle NQR from K along the straight line KR.

Hence the evolute of the cycloid OBO' is composed of the two semi-cycloids OK and KO'.

289. Length of Arc. To find the length of the arc OP (Fig. of Art. 284) we substitute in

$$s = \int \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx,$$

$$\frac{dy}{dx} = \cot \frac{\phi}{2}, \quad \text{and} \quad dx = 2\,a \sin^2 \frac{\phi}{2}\, d\phi.$$

We thus obtain

$$s = 2\,a\int_0^{\phi}\sin\frac{\phi}{2}\,d\phi = 4\,a\left(1-\cos\frac{\phi}{2}\right).$$

If $\phi = 2\,\pi$, we have for the entire arc, $OBO' = 8\,a$.

This result is also evident from the property of the evolute, from which

$$OQK = BK = 4\,a.$$

290. Area. To find the area between the curve and the axis of X, we substitute in

$$A = \int y\,dx,$$

$$y = a(1-\cos\phi), \qquad dx = a\,(1-\cos\phi)\,d\phi.$$

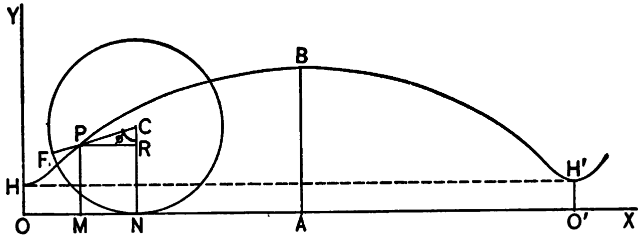
Thus we have for the entire area $OBO'A$,

$$A = a^2\int_0^{2\pi}(1-\cos\phi)^2\,d\phi = 3\,\pi a^2.$$

Hence this area is three times that of the generating circle.

TROCHOID

291. Definition and Equation. When a circle rolls along a straight line, a point on a radius or radius produced describes a curve called the *trochoid.*



As the circle, radius a, rolls on OX, the point P describes the trochoid $HPBH'$.

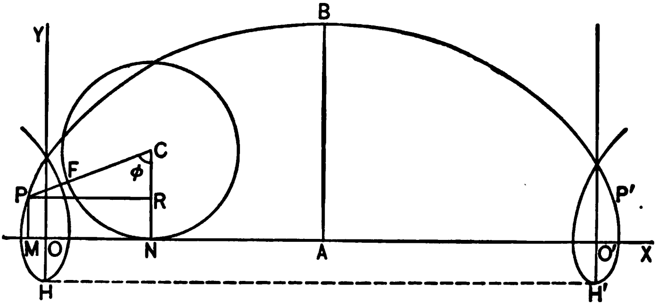
Let $PC = b$, and angle $FCN = \phi$.

$$ON = \text{arc } FN = a\phi.$$

Then
$$\left.\begin{aligned} x &= OM = ON - PR = a\phi - b\sin\phi, \\ y &= PM = CN - CR = a - b\cos\phi. \end{aligned}\right\} \quad \ldots \quad (1)$$

These are the parametric equations of the trochoid.

In the preceding figure $b < a$. If $b > a$, the generating point P is in the radius produced, and the trochoid is then of the form shown in the following figure, having loops about O, O', etc.



Equations (1) apply to both cases, when $b < a$, or when $b > a$.

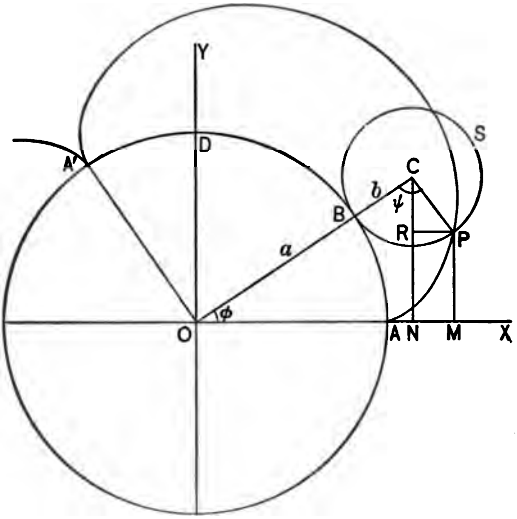
292. Area. The area included between one arch of the trochoid and the tangent at the lowest points, is found to be $\pi(2a + b)b$, when $b < a$, or when $b > a$. (See Ex. 16, p. 324.)

In the first figure this area is $HPBH'H$, in the second figure $HPBP'H'H$.

EPICYCLOID AND HYPOCYCLOID

293. Equation of Epicycloid. The epicycloid is the curve described by a point in the circumference of a circle, which rolls outside of a fixed circle.

Suppose the circle *BPS* to roll on the fixed circle *ADA'*, the point *P* describing the epicycloid *APA'*.



Let $OB=a,\ BC=b,\ BOA=\phi,\ BCP=\psi.$

Since the arcs *BA* and *BP* are equal, we have

$$a\phi=b\psi.$$

$$\begin{aligned} x=OM&=ON+RP\\ &=(a+b)\cos\phi+b\sin\left[\psi-\left(\frac{\pi}{2}-\phi\right)\right]\\ &=(a+b)\cos\phi-b\cos(\psi+\phi). \quad \ldots \quad (1)\end{aligned}$$

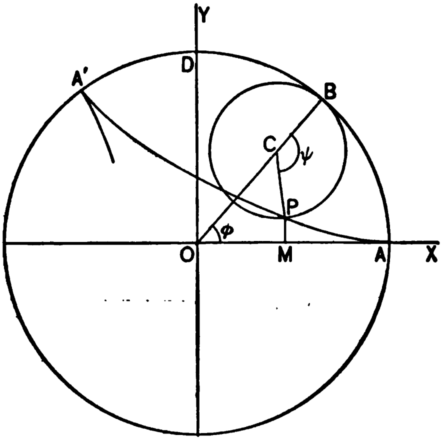
$$\begin{aligned} y=PM&=CN-CR\\ &=(a+b)\sin\phi-b\cos\left[\psi-\left(\frac{\pi}{2}-\phi\right)\right]\\ &=(a+b)\sin\phi-b\sin(\psi+\phi). \quad \ldots \quad (2)\end{aligned}$$

Substituting in (1) and (2), $\psi = \frac{a\phi}{b}$,

$$\left.\begin{aligned} x &= (a+b)\cos\phi - b\cos\frac{a+b}{b}\phi, \\ y &= (a+b)\sin\phi - b\sin\frac{a+b}{b}\phi. \end{aligned}\right\} \quad \dots\dots \quad (3)$$

294. Equation of Hypocycloid. The hypocycloid is the curve described by a point in the circumference of a circle, which rolls *inside* of a fixed circle.

Suppose the circle *BPS* to roll within the fixed circle *ADA'*, the point *P* describing the hypocycloid *APA'*.



Taking $OB = a$, $CB = b$, and $BOA = \phi$, the student will easily derive for the equations of the hypocycloid,

$$\left.\begin{aligned} x &= (a-b)\cos\phi + b\cos\frac{a-b}{b}\phi, \\ y &= (a-b)\sin\phi - b\sin\frac{a-b}{b}\phi. \end{aligned}\right\} \quad \dots \quad (1)$$

Notice that these equations may be obtained from (3) of the preceding article, by changing b into $-b$.

295. Cardioid. This is a special case of the epicycloid when $a=b$. Equations (3), Art. 293, then become

$$x=2a\cos\phi-a\cos 2\phi. \qquad (1)$$

$$y=2a\sin\phi-a\sin 2\phi. \qquad (2)$$

We may eliminate ϕ between these equations by first adding the squares of (1) and (2), giving

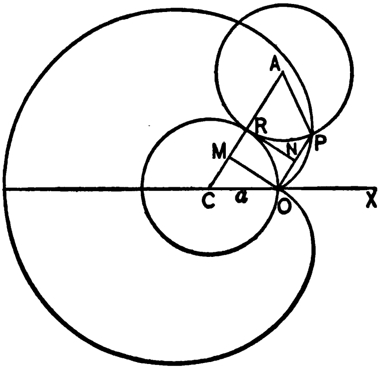
$$x^2+y^2=5a^2-4a^2\cos\phi. \qquad (3)$$

Substituting the value of $\cos\phi$ from (3) in (1),

$$(x^2+y^2-5a^2)(x^2+y^2-a^2)=8a^3(a-x),$$

which may be written $(x^2+y^2-3a^2)^2=4a^3(3a-2x)$.

But the equation of the cardioid is much more simply expressed and derived in polar coördinates, the origin being at O on the circumference of the fixed circle.



Here $OP=r$, angle $POX=\theta$. Evidently

angle $MCO=POX=\theta$.

$CO\cos MCO=CM=CR-ON$,

that is, $a\cos\theta=a-\frac{r}{2}$;

whence $r=2a(1-\cos\theta)$.

(See p. 170.)

296. Hypocycloid $a=2b$. Here equations (1), Art. 294, become

$$x=a\cos\phi, \qquad y=0,$$

the equations of the axis of X.

That is, when a circle rolls inside of a fixed circle of twice its diameter, a point on the circumference of the moving circle describes a diameter of the fixed circle.

297. Astroid. Another special case of the hypocycloid is $a = 4b$. This curve is sometimes called the *astroid.*

Here equations (1), Art. 294, become

$$x = \frac{3a}{4}\cos\phi + \frac{a}{4}\cos 3\phi = a\cos^3\phi,$$

$$y = \frac{3a}{4}\sin\phi - \frac{a}{4}\sin 3\phi = a\sin^3\phi.$$

Eliminating ϕ, $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$ (See p. 166.)

298. Radius of Curvature of Epicycloid. By differentiating (3) Art. 293, we have

$$\frac{dx}{d\phi} = (a+b)\left(\sin\frac{a+b}{b}\phi - \sin\phi\right) \quad . \; . \; . \; . \; . \; . \quad (1)$$

$$= 2(a+b)\sin\frac{a}{2b}\phi\cos\frac{a+2b}{2b}\phi. \quad . \; . \; . \quad (2)$$

$$\frac{dy}{d\phi} = (a+b)\left(-\cos\frac{a+b}{b}\phi + \cos\phi\right) \quad . \; . \; . \; . \quad (3)$$

$$= 2(a+b)\sin\frac{a}{2b}\phi\sin\frac{a+2b}{2b}\phi. \quad . \; . \; . \quad (4)$$

Therefore

$$\frac{dy}{dx} = \tan\frac{a+2b}{2b}\phi;$$

whence

$$\frac{d^2y}{dx^2} = \frac{a+2b}{2b}\sec^2\frac{a+2b}{2b}\phi\cdot\frac{d\phi}{dx} = \frac{a+2b}{4b(a+b)}\,\frac{\sec^3\frac{a+2b}{2b}\phi}{\sin\frac{a}{2b}\phi}.$$

Substituting in the formula for the radius of curvature, we find

$$\rho = \frac{\left(1+\tan^2\frac{a+2b}{2b}\phi\right)^{\frac{3}{2}}}{\sec^3\frac{a+2b}{2b}\phi}\cdot\frac{4b(a+b)}{a+2b}\sin\frac{a}{2b}\phi$$

$$= \frac{4b(a+b)}{a+2b}\sin\frac{a}{2b}\phi = \frac{4b(a+b)}{a+2b}\sin\frac{\psi}{2}. \quad . \; . \; . \; . \; . \; . \quad (5)$$

If $a=\infty$, the epicycloid becomes the cycloid, and

$$\text{Lim}_{a=\infty}\frac{a+b}{a+2b}=1.$$

Hence $$\rho=4b\sin\frac{\psi}{2}, \text{ as in Art. 287.}$$

299. Radius of Curvature of Hypocycloid. By changing b into $-b$ in (5) of the preceding article, we have for the radius of curvature of the hypocycloid, numerically,

$$\rho=\frac{4b(a-b)}{a-2b}\sin\frac{\psi}{2}.$$

300. Length of Arc. From (2) and (4), Art. 298, we have

$$\left(\frac{ds}{d\phi}\right)^2=\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2=4(a+b)^2\sin^2\frac{a}{2b}\phi.$$

Hence for the entire arch *APA'* (Fig. Art. 293), we have

$$s=2(a+b)\int_0^{\frac{2\pi b}{a}}\sin\frac{a}{2b}\phi d\phi=\frac{8(a+b)b}{a}.$$

For the hypocycloid, the length of one arch is

$$s=\frac{8(a-b)b}{a}.$$

301. Area between Curve and Fixed Circle. To find the area *APA'BA* (Fig. Art. 293), it is better to use polar coördinates r, θ.
The formula

$$A=\frac{1}{2}\int r^2d\theta,$$

will give the area *APA'OA*, and this, less the area of the sector *A'OA*, will be the required area.

Differentiating $$\frac{y}{x} = \tan\theta,$$

we have $$\frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta,$$

$$x\,dy - y\,dx = x^2\sec^2\theta\,d\theta = r^2\,d\theta.$$

From (3), Art. 293, and (1), (3), Art. 298, we find

$$x\,dy - y\,dx = (a+b)(a+2b)\left(1 - \cos\frac{a}{b}\phi\right)d\phi.$$

Therefore

$$\int r^2\,d\theta = (a+b)(a+2b)\int\left(1 - \cos\frac{a}{b}\phi\right)d\phi.$$

Hence

Area $$APA'OA = \frac{1}{2}(a+b)(a+2b)\int_0^{\frac{2\pi b}{a}}\left(1 - \cos\frac{a}{b}\phi\right)d\phi$$

$$= \frac{\pi b}{a}(a+b)(a+2b).$$

Subtracting the area of the sector $AOA' = \pi ab$, we have

Area $$APA'BA = \pi b\left[\frac{(a+b)(a+2b) - a^2}{a}\right] = \frac{\pi b^2(3a+2b)}{a}.$$

The corresponding area for the hypocycloid is

$$\frac{\pi b^2(3a-2b)}{a}.$$

CHAPTER XXXIV

INTEGRALS FOR REFERENCE

302. We give for reference a list of some of the integrals of the preceding chapters.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1}.$

2. $\int \frac{dx}{x} = \log x.$

3. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$

4. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}.$

EXPONENTIAL INTEGRALS

5. $\int a^x\,dx = \frac{a^x}{\log a}.$

6. $\int e^x\,dx = e^x.$

TRIGONOMETRIC INTEGRALS

7. $\int \sin x\,dx = -\cos x.$

8. $\int \cos x\,dx = \sin x.$

9. $\int \tan x\,dx = \log \sec x.$

10. $\int \cot x\,dx = \log \sin x.$

11. $\int \sec x\,dx = \log(\sec x + \tan x)$

$$= \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

12. $\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$

$$= \log \tan \frac{x}{2}.$$

13. $\int \sec^2 x\,dx = \tan x.$

14. $\int \operatorname{cosec}^2 x\,dx = -\cot x.$

15. $\int \sec x \tan x\,dx = \sec x.$

16. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x.$

17. $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x.$

18. $\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x.$

INTEGRALS CONTAINING $\sqrt{a^2 - x^2}$ (CHAP. XXV. AND ART. 227)

19. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}.$

20. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

21. $$\int\frac{dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\log\frac{x}{a+\sqrt{a^2-x^2}}.$$

22. $$\int\frac{dx}{x^2\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{a^2x}.$$

23. $$\int\frac{dx}{x^3\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{2\,a^2x^2}+\frac{1}{2\,a^3}\log\frac{x}{a+\sqrt{a^2-x^2}}.$$

24. $$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

25. $$\int x^2\sqrt{a^2-x^2}\,dx=\frac{x}{8}(2\,x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}.$$

26. $$\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{a^2\sqrt{a^2-x^2}}.\quad\text{(Art. 227.)}$$

27. $$\int(a^2-x^2)^{\frac{3}{2}}\,dx=\frac{x}{8}(5\,a^2-2\,x^2)\sqrt{a^2-x^2}+\frac{3\,a^4}{8}\sin^{-1}\frac{x}{a}.$$

INTEGRALS CONTAINING $\sqrt{x^2+a^2}$ (CHAP. XXV. AND ART. 227)

28. $$\int\frac{dx}{\sqrt{x^2+a^2}}=\log(x+\sqrt{x^2+a^2}).$$

29. $$\int\frac{x^2\,dx}{\sqrt{x^2+a^2}}=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\log(x+\sqrt{x^2+a^2}).$$

30. $$\int\frac{dx}{x\sqrt{x^2+a^2}}=\frac{1}{a}\log\frac{x}{a+\sqrt{x^2+a^2}}=\frac{1}{a}\log\frac{\sqrt{x^2+a^2}-a}{x}.$$

31. $$\int\frac{dx}{x^2\sqrt{x^2+a^2}}=-\frac{\sqrt{x^2+a^2}}{a^2x}.$$

32. $$\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{a+\sqrt{x^2+a^2}}{x}.$$

33. $$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}).$$

34. $$\int x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2+a^2}).$$

35. $$\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{x^2+a^2}}.$$

36. $$\int (x^2+a^2)^{\frac{3}{2}}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^4}{8}\log(x+\sqrt{x^2+a^2}).$$

INTEGRALS CONTAINING $\sqrt{x^2-a^2}$ (CHAP. XXV. AND ART. 227)

37. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}).$$

38. $$\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}).$$

39. $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

40. $$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}.$$

41. $$\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{x}{a}.$$

42. $$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right).$$

43. $$\int x^2\sqrt{x^2 - a^2}\, dx = \frac{x}{8}(2\,x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 - a^2}\right).$$

44. $$\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}} = -\frac{x}{a^2\sqrt{x^2 - a^2}}.$$

45. $$\int (x^2 - a^2)^{\frac{3}{2}}\, dx = \frac{x}{8}(2\,x^2 - 5\,a^2)\sqrt{x^2 - a^2} + \frac{3\,a^4}{8}\log\left(x + \sqrt{x^2 - a^2}\right)$$

INTEGRALS CONTAINING $\sqrt{2\,ax - x^2}$

46. $$\int \frac{dx}{\sqrt{2\,ax - x^2}} = \operatorname{vers}^{-1}\frac{x}{a}.$$

47. $$\int \frac{x\,dx}{\sqrt{2\,ax - x^2}} = -\sqrt{2\,ax - x^2} + a\operatorname{vers}^{-1}\frac{x}{a}.$$

48. $$\int \frac{dx}{x\sqrt{2\,ax - x^2}} = -\frac{\sqrt{2\,ax - x^2}}{ax}.$$

49. $$\int \sqrt{2\,ax - x^2}\, dx = \frac{x - a}{2}\sqrt{2\,ax - x^2} + \frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}.$$

50. $$\int x\sqrt{2\,ax - x^2}\, dx = -\frac{3\,a^2 + ax - 2\,x^2}{6}\sqrt{2\,ax - x^2} + \frac{a^3}{2}\operatorname{vers}^{-1}\frac{x}{a}$$

51. $$\int \frac{\sqrt{2\,ax - x^2}\, dx}{x} = \sqrt{2\,ax - x^2} + a\operatorname{vers}^{-1}\frac{x}{a}.$$

52. $$\int \frac{\sqrt{2ax-x^2}\,dx}{x^3} = -\frac{(2ax-x^2)^{\frac{3}{2}}}{3ax^3}.$$

53. $$\int \frac{dx}{(2ax-x^2)^{\frac{3}{2}}} = \frac{x-a}{a^2\sqrt{2ax-x^2}}.$$

54. $$\int \frac{x\,dx}{(2ax-x^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{2ax-x^2}}.$$

INTEGRALS CONTAINING $\pm ax^2+bx+c$

55. $$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}},$$

56. or $$= \frac{1}{\sqrt{b^2-4ac}}\log\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}.$$

57. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\log\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right).$$

58. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c}$$
$$-\frac{b^2-4ac}{8a^{\frac{3}{2}}}\log\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right).$$

59. $$\int \frac{dx}{\sqrt{-ax^2+bx+c}} = \frac{1}{\sqrt{a}}\sin^{-1}\frac{2ax-b}{\sqrt{b^2+4ac}}.$$

60. $$\int \sqrt{-ax^2+bx+c}\,dx$$
$$= \frac{2ax-b}{4a}\sqrt{-ax^2+bx+c} + \frac{b^2+4ac}{8a^{\frac{3}{2}}}\sin^{-1}\frac{2ax-b}{\sqrt{b^2+4ac}}.$$

OTHER INTEGRALS

61. $$\int \sqrt{\frac{a+x}{b+x}}\,dx \qquad \text{(Art. 233)}$$
$$=\sqrt{(a+x)(b+x)}+(a-b)\log\left(\sqrt{a+x}+\sqrt{b+x}\right).$$

62. $$\int \sqrt{\frac{a-x}{b+x}}\,dx=\sqrt{(a-x)(b+x)}+(a+b)\sin^{-1}\sqrt{\frac{x+b}{a+b}}.$$
(Art. 233)

63. $$\int \frac{dx}{a+b\sin x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}, \qquad \text{(Art. 231)}$$
$$\text{or }=\frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}.$$

64. $$\int \frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right), \qquad \text{(Art. 232)}$$
$$\text{or }=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}}.$$

65. $$\int e^{ax}\sin nx\,dx=\frac{e^{ax}(a\sin nx-n\cos nx)}{a^2+n^2}. \qquad \text{(Art. 217)}$$

66. $$\int e^{ax}\cos nx\,dx=\frac{e^{ax}(n\sin nx+a\cos nx)}{a^2+n^2}. \qquad \text{(Art. 217)}$$

If X is a rational polynomial in x,

67. $$\int e^{ax}X\,dx=\frac{e^{ax}}{a}\left(X-\frac{1}{a}\frac{dX}{dx}+\frac{1}{a^2}\frac{d^2X}{dx^2}-\frac{1}{a^3}\frac{d^3X}{dx^3}+\cdots\right).$$